Honours Analysis Skills
Example Presentation Template

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Uniform convergence

Definition

A sequence of functions $f_n \colon \mathbb{R} \to \mathbb{R}$ *converges uniformly* to a function $f \colon \mathbb{R} \to \mathbb{R}$ if for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $n \geq N$ implies

$$\sup_{x \in \mathbb{R}} |f_n(x) - f(x)| < \epsilon.$$

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Pointwise and uniform continuity

- Uniform convergence implies pointwise convergence
- Pointwise convergence does not imply uniform convergence

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Theorem

Let $f_n \colon \mathbb{R} \to \mathbb{R}$ be continuous and converge uniformly to $f \colon \mathbb{R} \to \mathbb{R}$. Then f is continuous.

Uniform convergence and continuity

Proof.

Let $x \in \mathbb{R}$ and let $\epsilon > 0$. There exists $N \in \mathbb{N}$ such that $n \geq N$ implies

$$\sup_{x \in \mathbb{R}} |f_n(x) - f(x)| < \frac{\epsilon}{3}. \tag{1}$$

There exists $\delta > 0$ such that

$$|f_N(x) - f_N(y)| < \frac{\epsilon}{3} \text{ whenever } |x - y| < \delta. \tag{2}$$

Then inequalities (1) and (2) imply that whenever $|x - y| < \delta$, we have

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon. \end{aligned}$$

□

People

This subject owes much to

Figure: Augustin-Louis Cauchy

Figure: Karl Weierstrass