1 Example

1. Prove naive set theory inconsistent

Solution:

Proof. Consider the set $S = \{X | X \notin X\}$

$$S \in S \iff S \notin S$$

. ■

2 Example 2

1. Formulate and prove DeMorgan's laws for arbitrary unions and intersections

Solution:

Proof. DeMorgan's laws for arbitrary unions can be stated as follows:

$$A - \bigcup_{b \in B} b = \bigcap_{b \in B} (A - b)$$

$$A - \bigcap_{b \in B} b = \bigcup_{b \in B} (A - b)$$

Consider an arbitrary $x \in A$. By the definition of a complement, $x \in A - \bigcap_{b \in B} b$ if and only if $x \notin \bigcup_{b \in B} b$. From there the proof follows by unfolding the definition of a union and letting quantifiers propagate out.

$$\begin{aligned} x \in A \wedge x \notin \bigcup_{b \in B} b &= x \in A \wedge \forall b \in B,\ x \notin b \\ &= \forall b \in B,\ x \in A \wedge x \notin b \\ &= \forall b \in B,\ x \in A - b \\ &= x \in \bigcap_{b \in B} (A - b) \end{aligned}$$

■

The second proof is omitted, as it's essentially identical to the first

2. Prove that $\mathcal{P}(\mathbb{N})$ and $\mathbb{R}$ have the same cardinality

Solution:

Proof. Instead of giving a bijection directly we will put both $\mathcal{P}(\mathbb{N})$ and $\mathbb{R}$ in bijection with $2^{\mathbb{N}}$. First we'll handle $\mathcal{P}(\mathbb{N})$. For any subset S of $\mathbb{N}$, we give the function

$$f(x) = \begin{cases} 1 & : x \in S \\ 0 & : x \notin S \end{cases}$$

Which has as inverse the function $g(f) = \{n \mid n \in \mathbb{N},\ f(n) = 1\}$.

For the bijection between $2^{\mathbb{N}}$ and $[0, 1)$, we'll make use of the fact that any $r \in [0, 1)$ can be written as an infinite binary expansion, $0.r_0r_1r_2$. We can thereby write any real number r as a function $f : \mathbb{N} \to 2$ by taking $f(n) = r_n$. We can go in the other direction by taking $r_n = f(n)$. ■