Solutions to in-class problems

College Geometry

Spring 2016

Theorem 3.1.7. If ℓ and m are two distinct, nonparallel lines, then there exists exactly one point P such that P lies on both ℓ and m.

Proof. (Selena Emerson) Let ℓ and m be two distinct, nonparallel lines. By definition of nonparallel there exists at least one point, P, that lies on both ℓ and m. Suppose there exists a different point, Q, be on both ℓ and m. By Axiom 3.1.3, there exists exactly one line on which the two distinct points lie. Since P and Q are on both ℓ and m, then ℓ and m are the same line. But ℓ and m are defined as being distinct lines which is a contradiction. Thus, there exists exactly one point P such that P lies on both ℓ and m.

□

Problem 3.2.3. Show that the taxicab metric defined in Example 3.2.11 is a metric(i.e., verify that the function ρ satisfies the three conditions in the definition of metric on page 339).

Proof. (Brianna Hillman) We want to show that the taxicab metric is a metric. To show this, we need to show $D(P,Q) = D(Q,P)$ for every P and Q, $D(P,Q) \geq 0$ for every P and Q, and that $D(P,Q) = 0$ if and only if $P = Q$. Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$. Take $P, Q \in R^2$. Then we have that $\rho((x_1, y_1), (x_2, y_2)) = \rho((x_2, y_2), (x_1, y_1))$

$\Rightarrow \mid x_2 - x_1 \mid + \mid y_2 - y_1 \mid = \mid x_1 - x_2 \mid + \mid y_1 - y_2 \mid.$

We know that this is equal because of the properties of the absolute values. Thus, the first condition of a metric is satisfied. Now, take a P, $Q \in R^2$. If $P = Q$, then

$\rho((x_1, y_1), (x_1, y_1)) = \mid x_1 - x_1 \mid + \mid y_1 - y_1 \mid = 0$. But, if $P \neq Q$, then

$PQ = \mid x_2 - x_1 \mid + \mid y_2 - y_1 \mid \geq 0$. Thus, $\rho((x_1, y_1), (x_2, y_2)) = \mid x_2 - x_1 \mid + \mid y_2 - y_1 \mid \geq 0$.

Therefore, the second condition of a metric is satisfied. Next we will prove the third condition.

($\Rightarrow$) Suppose $\rho(P,Q) = 0$. Then, $\mid x_2 - x_1 \mid + \mid y_2 - y_1 \mid = 0$. So $\mid x_2 - x_1 \mid = 0$ and $\mid y_2 - y_1 \mid = 0$.

Therefore, $x_2 = x_1$ and $y_2 = y_1$ or $P = Q$. ($\Leftarrow$) Now, suppose $P = Q$. So, $P = Q = (x_1, y_1)$.

Then, $D(P, Q) = \rho((x_1, y_1), (x_1, y_1))$

$\Rightarrow = \mid x_1 - x_1 \mid + \mid y_1 - y_1 \mid$

$\Rightarrow = 0 + 0$

$\Rightarrow = 0.$

Therefore, the third condition of a metric is satisfied and the taxicab metric is a metric.

$\square$

Problem 3.2.7. Find all point (x, y) in $\mathbb{R}^2$ such that $\rho((0,0),(x,y)) = 1$, where ρ is the taxi cab metric. Draw a sketch in the Cartesian plane. (This shape might be called a "circle" in the taxicab metric.)

Proof. (Victoria Krohn)

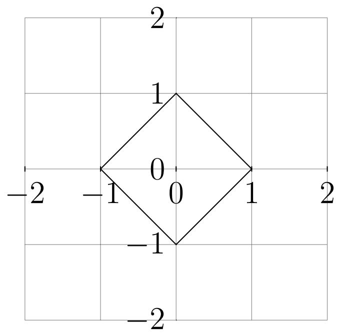
$$\rho((0,0),(x,y)) = |x - 0| + |y - 0| = 1$$

$(x, y) = (1, 0)$ $\quad$ $\rho((0,0),(1,0)) = |1 - 0| + |0 - 0| = 1$

$(x, y) = (0, 1)$ $\quad$ $\rho((0,0),(0,1)) = |0 - 0| + |1 - 0| = 1$

$(x, y) = (-1, 0)$ $\quad$ $\rho((0,0),(-1,0)) = |(-1) - 0| + |0 - 0| = 1$

$(x, y) = (0, -1) \qquad \rho((0,0),(0,-1)) = |0 - 0| + |(-1) - 0| = 1$



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Problem 3.2.21. Let A and B be two distinct points. Prove that $\overline{AB} = \overline{BA}$.

Proof. (Brianna Hillman) Assume A and B are distinct points. Note that

$\overline{AB} = \{A, B\} \cup \{P \mid A \star P \star B\}$ and $\overline{BA} = \{B, A\} \cup \{P \mid B \star P \star A\}$. Let $x \in \overline{AB}$.

So $x \in \{A, B\} \cup \{P \mid A \star P \star B\}$

$\Rightarrow x \in \{A, B\}$ or $x \in \{P \mid A \star P \star B\}$

In the case where $x = A$ or $x = B$, then $x \in \overline{BA}$. In the case where $x \neq A$ and $x \neq B$, x is in between A and B. Therefore, $Ax + xB = AB$ by the definition of between. Since $AB = Ax + xB$, then we have that

$\Rightarrow AB = xB + Ax$

$\Rightarrow AB = Bx + xA$ by Theorem 3.2.7

Since AB = Bx + xA, then x is between B and A and hence, $\overline{AB} \subseteq \overline{BA}$. Similarly, if we take an $x \in \overline{BA}$, we conclude that $BA = Ax + xB$ and x is between A and B. Thus, $\overline{BA} \subseteq \overline{AB}$. Therefore, $\overline{AB} = \overline{BA}$. □

Theorem 3.3.12. (Pasch's Axiom) Let $\triangle ABC$ be a triangle and let l be a line such that none of A, B, and C lies on l. If l intersects $\overline{AB}$, then l also intersects BC or AC.

Proof. (Katelynn Gordon) Let points A, B, and C form a triangle △ABC and let l be a line that does not go through a vertex of △ABC. Then l has to go in and come out of the triangle. Then there are three options: $C \in l$, $C \in H_1$, or $C \in H_2$. If C $\in H_1$, then since $B \in H_2$, $\overline{BC} \cap l \neq \emptyset$ (3.3.4). Or if $C \in H_2$, then since $A \in H_1$ $\overline{AC} \cap l \neq \emptyset$. In either case, l crosses $\overline{AC}$ or $\overline{BC}$. □

Problem 3.3.3. Let l be a line and let H be one of the half-planes bounded by l. Prove that $H \cup l$ is a convex set.

Proof. (Victoria Krohn) Let l be a line and H be a half-plane bounded by l. We need to show that that $H \cup l$. Let points A, B exist such that they construct $\overline{AB}$. We consider 3 cases, when $\overline{AB} \cup H$, when $\overline{AB} \cup l$ and when $A \cup l$ and $B \cup H$. When $\overline{AB} \cup H$ by the Plane Separation Postulate $\overline{AB}$ is in $H \cup l$. When $\overline{AB} \cup l$ by the Incidence Postulate they are in $H \cup l$. Let $A \in l$ and $B \in H$ so $\overrightarrow{AB} \in H$ by the Ray Theorem, therefore $H \cup l$. Thus $H \cup l$ is convex. □

Problem 3.3.5. Suppose $\triangle ABC$ is a triangle and l is a line such that none of the vertices A, B, or C lies on l. Prove that l cannot intersect all three sides of $\triangle ABC$. Is it possible for a line to intersect all three sides of a triangle?

Proof. (Victoria Krohn) Assume line l intersects a side of a triangle $\triangle ABC$. Without loss of generality, l intersects with side $\overline{AB}$. This creates 2 half-planes, H_1, H_2 with $A \in H_1$ and $B \in H_2$. By Pasch's Axiom, l must exit $\triangle ABC$ through another side. Thus $C \in H_1$ or $C \in H_2$. Therefore, l cannot cross all 3 sides of the triangle. □

Problem 3.5.1. Prove: If $l \perp m$, then l and m contain rays that make four different right angles.

Proof. Victoria Krohn Let A, B, C, D, E be distinct points, such that $A, B, E \in$ line l and $D, A, C \in$ line m with $m \perp l$ intersecting at point A. We need to show that $\mu\angle(BAC)$,

$\mu\angle(CAE)$, $\mu\angle(DAE)$, and $\mu\angle(DAB)$ equal $90°$. By definition of perpendicular, $\angle BAC$ is a right angle, thus $\mu\angle(BAC) = 90°$. Angles $\angle DAB$ and $\angle BAC$ are a linear pair by the opposite rays $\overrightarrow{AD}$ and $\overrightarrow{AC}$ (definition of linear pair). By the Linear Pair Theorem, $\mu\angle(BAC)+\mu\angle(DAB) = 180°$. So, $90°+\mu\angle(DAB) = 180°$, $\mu\angle(DAB) = 90°$. By definition of right angle, $\angle DAB$ is a right angle. Similarly for angles $\angle DAB$ and $\angle DAE$, and $\angle DAE$ and $\angle CAE$. □

Problem 3.5.2. Prove existence and uniqueness of a perpendicular to a line at a point on the line (Theorem 3.5.9).

Proof. (Brianna Hillman) Let ℓ be a line where P and Q are two distinct points on ℓ. By the Angle Construction Postulate, there exists a unique ray $\overrightarrow{PA}$ such that A is in one half-plane bounded by ℓ and $\mu(\angle APQ) = 90°$. Then, we can extend ray $\overrightarrow{PA}$ to a line by the Incidence Postulate. So, there exists exactly one line $\overleftrightarrow{PA}$ such that P lies on $\overleftrightarrow{PA}$ and $\overleftrightarrow{PA} \perp \ell$. □

Problem 3.5.5. Restate the Vertical Angles Theorem (Theorem 3.5.13) in if-then form. Prove the theorem.

If the angles are vertical, then they are congruent.

Proof. (Selena Emerson) Let $\angle BAC$ and $\angle DAE$ be vertical angles with A being the intersection of $\overleftrightarrow{BE}$ and $\overleftrightarrow{CD}$. Then $\angle BAC$ and $\angle CAE$ are linear pairs, making $\mu\angle(BAC) + \mu\angle(CAE) = 180°$ by the linear pair theorem. The same is true for $\angle CAE$ and $\angle DAE$. Thus, $\mu\angle(BAC) + \mu\angle(CAE) = \mu\angle(CAE) + \mu\angle(DAE)$ so $\mu\angle(BAC) = \mu\angle(DAE)$. Hence, $\angle BAC \cong \angle DAE$. □

Problem 3.5.6. Prove the following coverse of the Vertical Angles Theorem: If A, B, C, D, and E, are points such that A*B*C, D and E are on opposite sides of $\leftrightarrow AB$, and $\angle DBC \cong \angle ABE$, then D, B, and E are collinear.

Proof. Katelynn Gordon $BWOC$ Let If A, B, C, D, and E, are points such that A*B*C, D and E are on opposite sides of $\leftrightarrow AB$, and $\angle DBC \cong \angle ABE$. Then without loss of generality let $\overrightarrow{EB}$ be between $\angle ABF$. Then $\angle DBC \cong \angle ABF$ by the Veritcal Angles Theorem. Then $\mu(\angle ABE) + \mu(\angle EBF)$ should equal $\mu(\angle DBC)$ by angle addition. However, we assumed $\angle ABE \cong \angle DBC$ from the problem. So $\mu(\angle EBF)$ would have to be $0°$ so E, B, and D would have to be on the same line. □

Theorem 3.6.5 (Isosceles Triangle Theorem). The base angles of an isosceles triangle are congruent.

Proof. (Brianna Hillman) Let $\triangle ABC$ be a triangle such that $\overline{AB} \cong \overline{AC}$. We must prove that $\angle ABC \cong \angle ACB$. Let D be a point in the interior of $\angle BAC$ such that $\overrightarrow{AD}$ is the bisector of $\angle BAC$ (Theorem 3.4.7). There is a point E at which the ray $\overrightarrow{AD}$ intersects the segment $\overline{BC}$ (Crossbar Theorem). Then $\triangle BAE \cong \triangle CAE$ by SAS and so $\angle ABE \cong \angle ACE$. Thus $\angle ABC \cong \angle ACB$. □

Problem 3.7.1. Check that the trivial geometry containing just one point and no lines satisfies all the postulates for neutral geometry except the Existence Postulate. Which parallel postulate is satisfied by this geometry?

Proof. (Brianna Hillman) The Ruler Postulate and Incidence Postulate are vacuously true because there is only one point in trivial geometry. The Plane Separation Postulate is vacuously true because there are no lines in trivial geometry. The Protractor Postulate is vacuously true because there are no lines or rays to make angles. Finally, the Side-Angle-Side Postulate is vacuously true as well because there are no segments in trivial geometry to make triangles. None of the parallel postulates are satisfied in trivial geometry because there is only one point. □

Problem 4.2.1. Prove the Converse to the Isosceles Triangle Theorem (Theorem 4.2.2).

Proof. (Brianna Hillman) Let $\triangle ABC$ be a triangle such that $\angle ABC \cong \angle ACB$. We want to show that $\overline{AB} \cong \overline{AC}$. By the Existence and Uniqueness of Perpendiculars, we can drop a perpendicular from A to some point P on $\overline{BC}$. Then, since $\angle ABC \cong \angle ACB, AP = AP$, and $\angle APB \cong \angle APC$, $\triangle ABP \cong \triangle APC$ by AAS. Therefore, $\overline{AB} \cong \overline{AC}$. □

Problem 4.2.4. Prove the HYpotenus-Leg Theorem

Proof. (Katelynn Gordon) Let $\triangle ABC$ be a triangle such that $\mu(\angle BAC)$=90°. Then extend $\overrightarrow{AC}$ so that $\overline{CA} \cong \overline{AF}$. Now forming $\overline{FB}$ creates two triangles $\triangle ABC$ and $\triangle ABF$. $\triangle ABC \cong \triangle ABF$ by SAS so $\overline{FB} \cong \overline{CB}$. □

Problem 4.2.5. Prove that it is possible to construct a congruent copy of a triangle on a given base (Theorem 4.2.6).

Proof. (Victoria Krohn) Let $\triangle ABC$ be a triangle, $\overline{D}E$ is a segment such that $\overline{D}E \cong \overline{A}C$, and H is a half-plane bounded by $\overleftrightarrow{D}E$. We need to show $\triangle ABC$ is congruent to another triangle contructed from $\overleftrightarrow{DE}$. Let point G be in H. By the Angle Construction Postulate, construct $\overrightarrow{DG}$ such that $\angle BAC \cong \angle GDE$. By the Point Construction Postulate, let point F lie on $\overrightarrow{DG}$ such that $\overline{AB} \cong \overline{DF}$. Form $\triangle DFE$. By SAS, $\triangle ABC \cong \triangle DFE$ because $\overline{AB} \cong \overline{DF}$, $\angle BAC \cong \angle GDE$, and $\overline{DE} \cong \overline{AC}$. □

Problem 4.3.7. Prove that the shortest distance from a point to a line is measured along the perpendicular (Theorem 4.3.4)

Proof. (Katelynn Gordon) Let there be a line ℓ such that F, $R \in \ell$ and $F \neq R$. Then let P be a point such that $P \notin \ell$. Then we can drop a perpendicular from P to ℓ that goes through F. We can also connect P to R in order to form $\triangle\ PFR$. We know that $\angle\ PFR$ is a right angle since it is formed by a perpendicular. We then know that $\triangle\ PFR$ is a right triangle and therefore we know that the hypotenuse which is across from the right angle of a right triangle in the longest side of the triangle by the Scalene Inequality. Therefore, the perpedicular is shorter than the hypotenuse of the right triangle. □

Problem 4.3.8. Prove the Pointwise Characterization of Angle Bisectors (Theorem 4.3.8)

Let A and B be distinct points. A point P lies on the perpendicular bisector of $\overline{AB}$ if and only if $PA = PB$.

Proof. (Victoria Krohn) $\Leftarrow$ Assume $d(P, \overleftrightarrow{AB}) = d(P\overleftrightarrow{AC})$. By the Existence and Uniqueness of Perpendiculars, drop a perpendicular from P to point N on $\overrightarrow{AC}$ and to point M on $\overrightarrow{AB}$. By the definiton of perpendicular, $\mu(\angle PNA) = 90°$ and $\mu(\angle PMA) = 90°$. By definiton of congruce, $\angle PNA = \angle PMA$. By the Hypotenuse - Leg Theorm, $\triangle ANP \cong \triangle AMP$ because $\overline{AP} \cong \overline{AP}$, $\angle PNA = \angle PMA$, and $\overline{NP} \cong \overline{MP}$ (by the assumptions). By definiton of congruent triangles, $\angle PAN \cong \angle PAM$, therefore $\overrightarrow{AP}$ is an angle bisector of $\angle CAB$.

$\Rightarrow$ Assume $\overrightarrow{AP}$ is an angle bisector of $\angle CAB$. Let N be a point on $\overrightarrow{AC}$ and M be a point on $\overrightarrow{AB}$. By the Existence and Uniqueness of Perpendiculars, drop a perpendicular from P to N and from P to M. By the definiton of perpendicular, $\mu(\angle PNA) = 90°$ and $\mu(\angle PMA) = 90°$. By definiton of congruce, $\angle PNA \cong \angle PMA$. By Angle-Angle-Side, $\triangle ANP \cong \triangle AMP$ because $\overline{AP} \cong \overline{AP}$, $\mu(\angle PAN) = \mu(\angle PAM)$ (by definition of Angle Bisectors) so $\angle PAN \cong \angle PAM$ (by defintion of congruence), and $\angle PNA \cong \angle PMA$. Thus, by definiton of congruent triangles, $\overline{NP} \cong \overline{MP}$. $\square$

Problem 4.4.3. Prove Corollary 4.4.8 If l, m and n are three lines such that $m \perp l$ and

$n \perp n$, then either $m = n$ or $m \parallel n$.

Proof. (Katelynn Gordon) Assume $m \neq n$. Then we want to show that $m \parallel n$. Then know that if $l \perp m$ and $l \perp n$ then l is a transversal. Then by the definition of perpendicular, we know that l crosses m and n at $90°$. We can then see that have corresponding congruent angles so it follows that $m \parallel n$. $\square$

Problem 4.6.6. Prove that a quadrilateral is convex if the diagonals have a point in common (the remaining part of Thm 4.6.8).

Proof. (Connor Lowman) Let $\square ABCD$ be a quadrilateral and let diagonals $\overrightarrow{AC} \cap \overrightarrow{BD}$ at interior point E. So $D * E * B$ and $A * E * C$. Then E is in the intersection of the half-plane formed by $\overleftrightarrow{DC}$ and A, and the half-plane formed by $\overleftrightarrow{AD}$ and C. Thus E is in the interior of $\angle ADC$. By the Ray Theorem, E and B are on the same side of $\overleftrightarrow{DC}$. Therefore, B is in the interior of $\angle ADC$. $\square$

Problem 4.6.10. Let $\square ABCD$ be a convex quadrilateral. Prove that each of the following conditions implies that $\square ABCD$ is a parallelogram.

a. $\triangle ABC \cong \triangle CDA$

Proof. (Connor Lowman) Let $\square ABCD$ be a convex quadrilateral such that $\triangle ABC \cong \triangle CDA$. Notice that $\overleftrightarrow{AC}$ is a transversal cutting through $\overleftrightarrow{AB}$ and $\overleftrightarrow{DC}$. Since $\triangle ABC \cong \triangle CDA$, $\angle ACB \cong \angle DAC$ and $\angle DCA \cong \angle BAC$. Thus, by Alternate Interior Angles Theorem, $\overleftrightarrow{AB} \| \overleftrightarrow{DC}$. Notice $\overleftrightarrow{AC}$ is also a transversal cutting through $\overleftrightarrow{AD}$ and $\overleftrightarrow{BC}$. Again, angles $\angle ACB$,$\angle DAC$ and $\angle DCA$,$\angle BAC$ are alternate interior angles. Similarly, by Alternate Interior Angles Theorem, $\overleftrightarrow{AD} \| \overleftrightarrow{BC}$. Therefore, $\square ABCD$ is a parallelogram. □

b. $AB = CD$ and $BC = AD$.

Proof. (Connor Lowman) Let $\square ABCD$ be a convex quadrilateral such that $AB = CD$ and $BC = AD$. Consider diagonal $\overline{AC}$. Then by SSS, $\triangle ABC \cong \triangle CDA$. Therefore, by proof from part a, $\square ABCD$ is a parallelogram. □

d. The diagonals $\overline{AC}$ and $\overline{BD}$ share a common midpoint.

Proof. (Kathryn Bragwell) Let $\square ABCD$ be a convex quadrilateral. Let E be a common midpoint $\overline{AC}$ and $\overline{BD}$ share. Thus $AE = EC$ and $DE = CE$. Notice $\angle BEC$ and $\angle AED$

are vertical angles thus $\angle BEC \cong \angle AED$ also $\angle CED \cong \angle BED$ because they are vertical angles also. Thus $\triangle ABE \cong \triangle CDE$ by ASA and $\triangle CBE \cong \triangle AED$ by ASA. Since $\overline{AC}$ is a transversal of $\overline{AB}$ and $\overline{DC}$, and $\angle BAD \cong \angle ECD$ then $\overline{AB} \parallel \overline{DC}$. Similarly $\overline{BC} \parallel \overline{AD}$.

□

End material for Exam 1 / Begin material for Exam 2.

Problem 5.1.5. Properties of 60-60-60 and 30-60-90 triangles. An equilateral triangle is one in which all 3 sides have equal lengths. (a) Prove that a Euclidean triangles is equilateral if and only if each of its angles measures 60°.

Proof. $\Longleftarrow$ Let $\triangle ABC$ be a triangle such that all angles are the same. By the Angle Sum Theorem $\angle ABC + \angle BCA + \angle CAB = 180, 180/3 = 60$ thus, all 3 angles are 60°. By the Converse to the Isosceles Triangle Theorem, $\mu(\angle CAB) = \mu(\angle BCA)$ so $\overline{AB} \cong \overline{BC}$ and $\mu(\angle ABC) = \mu(\angle CAB)$ so $\overline{BC} \cong \overline{AC}$. Therefore, $\overline{AB} \cong \overline{BC} \cong \overline{AC}$. $\Longrightarrow$ Assume all sides of $\triangle ABC$ are congruent. By the Isosceles Triangle Theorem, base angles $\mu(\angle CAB) =$

$\mu(\angle BCA)$, because sides $\overline{AB} \cong \overline{BC}$ and base angles $\mu(\angle ABC) = \mu(\angle CAB)$ because $\overline{BC} \cong \overline{AC}$. Therefore, $\mu(\angle ABC) = \mu(\angle CAB) = \mu(\angle BCA)$. □

(b)Prove that there is an equilateral triangle in Euclidean geometry.

Proof. Let A, B, C, D be 4 distinct points such that $\overline{AB}$ creates 2 half planes and that point D is in one of the half planes, to construct $\overrightarrow{AD}$ such that $\mu(\angle DAB) = 60°$ (by the Angle Construction Postulate). Let point C lie on $\overrightarrow{AD}$ such that $A * C * D*$ and that $\overline{AC} \cong \overline{AB}$ (by the Point Construction Postulate). Let $\overline{CB}$ form to construct $\triangle ABC$. By the Isosceles Triangle Theorem, base angles $\mu(\angle ACB) = \mu(\angle ABC)$ because $\overline{AC} \cong \overline{AB}$. By the Angle Sum Postulate, $\angle ABC + \angle ACB + \angle CAB = 180$, $\mu(\angle ACB) = \mu(\angle ABC)$ so $2(\angle ABC) + \angle CAB = 180$ Thus, $\angle CAB = 180 - 2(\angle ABC)$, so $180/3 = 60°$ and $\mu(\angle ABC) = \mu(\angle CAB) = \mu(\angle ACB)$. By part a, all sides are equal thus $\triangle ABC$ is an equilateral triangle.

□

(c) Split an equilateral triangle at the midpoint of one side to prove that there is a triangle whose angles measure 30°, 60°, and 90°.

Proof. Let $\triangle ABC$ be an equilateral triangle and let point D be the midpoint □

Problem 5.3.2. Prove the SAS Similarity Criterion (Theorem 5.3.3).

Proof. (Katelynn Gordon) Let $\triangle ABC$ with $C' \in \overline{AC}$ and $\triangle DEF$ be triangles such that $\overline{AC'} \cong \overline{DF}$, $\angle CAB \cong \angle FDE$, and AB/AC=DE/DF. Now by the Incidence Postulate, form a parallel line m to $\overline{CB}$ through point C'. Then m crosses $\overline{AB}$ at point B'. Similary form a parallel line to m called l through point A. Then $\angle BAC \cong \angle BAC$ since it is the same angle, $\angle AB'C \cong \angle ABC$ by properties of parallel lines, and $\angle ACB \cong \angle AC'B'$ by parallel line properties. So $\triangle ABC \cong \triangle AB'C'$ by since all of their angles are congruent. By the Parallel Projection Postulate AB'/AB=AC'/AC. Then AB'AC/AC=ABAC'/AC. It follows that AB'/AC'=ABAC'/AC. Then AB'/AC'=AB/AC. From the assumption, AB/AC=DE/DF=AB'/AC'. Then AB'/AC=DE/DF, and AB'=DE. so $\triangle AB'C' \cong \triangle DEF$ by SAS and then $\triangle ABC \sim \triangle DEF$. □

Problem 5.4.3. Prove the converse to the Pythagorean Theorem (Theorem 5.4.5)

Proof. (Kathryn Bragwell) Let $\triangle ABC$ be triangle such that $a^2+b^2=c^2$. By the angle construction postulate let $\angle DEF=90°$. By the ruler postulate let $EG=AC$. Since AC is across from the angle with the vertex B it is b according to notation. Thus $EG=AC=b$. Now, by the ruler postulate let EA=CB. Since CB is across from the angle with the vertex A it is a. Thus EA=CB=a. Thus we have a right triangle $\triangle HEG$. Let GH=d. So $\triangle HEG$ has a relationship of a^2+b^2=d^2 by the Pythagorean Theorem. Since a^2+b^2=c^2 and a^2+b^2=d^2, then c^2=d^2, $\sqrt{c}$=$\sqrt{d}$, c=d. Thus $\triangle ABC \cong \triangle HEG$ by SSS. Hence if $\triangle ABC$ is a triangle such that a^2+b^2=c^2, then $\angle BCA$ is a right angle. □

Problem 7.2.5. Let $\square ABCD$ be Euclidean parallelogram. Choose one side as a base and define the corresponding height for the parallelogram. Prove that the area of the parallelogram is the length of the base times the height.

Proof. (Kathryn Bragwell) Let $\square ABCD$ be a parallelogram. Drop a perpendicular from $\overline{DC}$ to $\overline{AB}$ at D. Let E be the foot of the perpendicular. Drop another perpendicular from $\overline{AB}$ to $\overline{DC}$ at B. Let F be the foot of the perpendicular. Thus there exist a right triangle $\triangle DEA$ and a right triangle $\triangle CFB$. Since $\overline{DE}$ and $\overline{FB}$ is perpendicular to $\overline{DF}$ and $\overline{BF}$, $\square BEFD$ is a rectangle by the definition of rectangles. By theorem 7.2.3 $\alpha\triangle DEA$ = 1/2(BE)(ED) and $\alpha\triangle CFB$ = 1/2(CF)(FB). By the Euclidean Area Postulate $\alpha\square BEFD$=(EB)(FB). Since $\triangle DEA$, αCFB, and $\square BEFD$ exist within $\square ABCD$ we can add their areas. Thus

$\alpha\square ABCD$=1/2(AE)(ED)+1/2(CF)(FB)+(EB)(FB). Since $\square BEFD$ is a rectangle it is also a parallelogram , therefore by properties of parallelograms we know $\overline{FB} \cong \overline{EB}$. Thus $\alpha\square ABCD$ =1/2(AE)(FB)+1/2(CF)(FB)+(EB)(FB). Likewise, since $\square ABCE$ is a parallelogram $\overline{AD} \cong \overline{BC}$. Now consider $\triangle AED$ and $\triangle CFB$, $\overline{ED} \cong \overline{FB}$, $\angle AEB \cong \angle BFC$, and $\overline{AD} \cong \overline{BC}$ therefore they are congruent by the hypotenuse leg theorem. Hence $\overline{AE} \cong \overline{FC}$. Thus $\alpha\square ABCD =$ 1/2 (AE)(FB)+1/2(AE)(FB)+(EB)(FB)=(AB)(FB)+(EB)(FB)=(FB)(AB+EB). Since $\overline{FB}$ is one of the perpendiculars we originally droped it is the height of $\square ABCD$ by the definition of height. AE+EB is the length of $\overline{AB}$ this is the base by the definition of base. Thus the area of $\square ABCD$ is base times height,

$\square$