Rectangular Circumhyperbolae

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Abstract

This paper deals with the Euclidean properties of rectangular circumhyperbolae with respect to a triangle using as little analytic treatment as possible. Familiarity with projective geometry, specifically ideal points, conic sections and Pascal's Theorem, is assumed.

Contents

1 Introduction to Circumconics

In this paper, the symbol $\measuredangle$ represents a directed angle modulo π.

Theorem 1. *The isogonal conjugate l^* of a line l in the plane of $\triangle ABC$ is a circumconic of $\triangle ABC$.*

Proof. We use homogenous barycentric coordinates. Let the line be $l \equiv ux+vy+wz=0$. Isogonal conjugation maps $P(x:y:z) \mapsto P^*(\frac{a^2}{x}:\frac{b^2}{y}:\frac{c^2}{z})$. Therefore the line l is mapped to $l^* \equiv \frac{ua^2}{x}+\frac{vb^2}{y}+\frac{wc^2}{z}=0 \equiv ua^2yz+vb^2zx+wc^2xy=0$, which is a second-degree curve and hence a conic. The reason it passes through the vertices is because a sequence of points on l converging to $l \cap BC$ have isogonal conjugates converging to A. Because isogonal conjugation is a continuous mapping, continuity ensures that $A \in l^*$. The other vertices also lie on l^* by symmetry. □

Theorem 2. *The isogonal conjugate of the ideal line is the circumcircle Ω of $\triangle ABC$.*

Proof. The ideal line $l_\infty \equiv x+y+z=0$ is mapped to $a^2yz+b^2zx+c^2xy=0$, which is nothing but $\Omega \equiv (ABC)$. □

Aliter. This theorem can also be proved by angle chasing. We show that the isogonal conjugate of a point P is an ideal point iff $P \in \Omega$. For that, let r_a, r_b and r_c be the lines isogonal to AP, BP and CP with to respect the corresponding vertices.
First assume that $P \in \Omega$. Then $\measuredangle(AB, r_a) = \measuredangle PAC = \measuredangle PBC = \measuredangle(AB, r_b)$, hence $r_a \parallel r_b$. By symmetry, r_c is also parallel to these lines and hence these concur at a point at infinity. For the converse, assume that $r_a \parallel r_b \parallel r_c$. Using essentially the same argument, $\measuredangle PAC = \measuredangle(AB, r_a) = \measuredangle(AB, r_b) = \measuredangle PBC \implies P \in \Omega$.

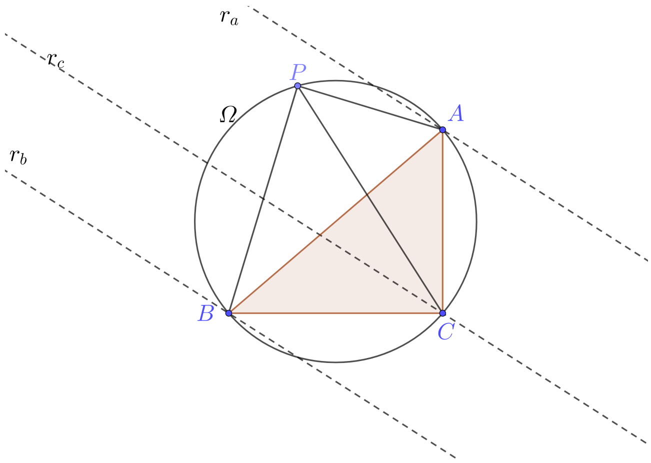

Figure 1: $\Omega^* \equiv l_\infty$

□

Corollary 2.1. *The nature of the circumconic l^* may be determined by counting the number of intersections of l with $\Omega \equiv (ABC)$. In particular, l^* is an ellipse, parabola or hyperbola according to whether l meets Ω in 0, 1 or 2 points respectively.*

Theorem 3. *l^* is a rectangular hyperbola iff l is a diameter of Ω. Equivalently, l^* is a rectangular hyperbola iff $H \equiv O^* \in l^*$.*

Proof. Suppose that $l \cap \Omega = \{X_1, X_2\}$. Then l^* is a hyperbola with points at infinity $Y_1 = X_1^*$ and $Y_2 = X_2^*$. By isogonal conjugates, $\measuredangle Y_1AY_2 = -\measuredangle X_1AX_2$ and hence the angle between the asymptotes of l^* is the angle subtended by X_1X_2 at Ω. In particular, the asymptotes are perpendicular iff X_1X_2 is a diameter of Ω. □

2 Revisiting Wallace-Simson Lines

Since we will need the discussion of Wallace-Simson lines in the following article, it is worth revising their properties.

Theorem 4. *Let l_P denote the Simson line of $P \in \Omega \equiv (ABC)$ with respect to $\triangle ABC$, and let H denote the orthocenter of $\triangle ABC$. Then l_P bisects PH, and this point of bisection lies on the nine-point circle Ω_9 of $\triangle ABC$.*

Proof. This proof can be found in [1].
Let X, Y and Z be the feet of perpendiculars from P to BC, CA and AB respectively. By definition, $X, Y, Z \in l_P$. Let AH meet Ω again in $H' \neq A$ and let PX meet Ω again in $K' \neq P$. Let K be the orthocenter of $\triangle PBC$. Then, we know that $K'H'$ is the image of KH in BC. Let $L \in l_P \cap AH$. Then $LA \parallel XK'$ and $\measuredangle AK'P = \measuredangle ABP = \measuredangle ZBP = \measuredangle ZXP$ where the last equality follows because $PZXB$ is cyclic with diameter PB. $\therefore AK' \parallel l_P \equiv LX \implies ALXK'$ is a parallelogram.
Because $AH \parallel PK$ and $AH = PK = 2R\cos A$, $AHKP$ is also a parallelogram. Consequently, $LH \parallel PX$ along with $LH = LA + AH = XK' + PK = KX + PK = PX$ implies that $PLHX$ is also a parallelogram. Therefore, $LX \equiv l_P$ bisects PH. Moreover, because the homothety $\mathbb{H}(H, \frac{1}{2})$ maps Ω to Ω_9, the midpoint of PH also lies on Ω_9. $\square$

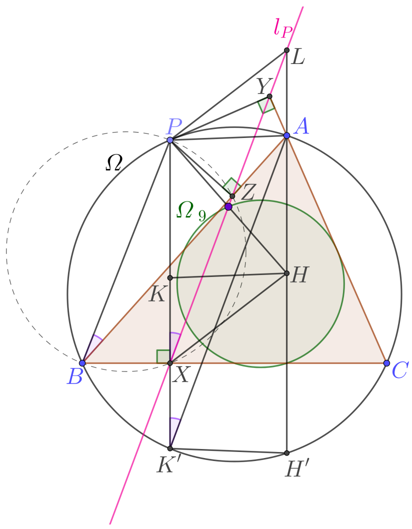

Figure 2: Simson line bisects the segment PH

Aliter. This proof is due to Ross Honsberger, and was taken from [2].
With notation as in the previous proof, let D be the foot of altitude from A and let $E \in PH' \cap BC$. Further, let M be the midpoint of PE. Since the triangles $\triangle HEH'$ and $\triangle XME$ are isosceles, $\measuredangle MXE = \measuredangle XEM = \measuredangle CEH' = \measuredangle HEC$, we get that $MX \parallel HE$. But on the other hand, becuase $PYCX$ is cyclic, $\measuredangle YXC = \measuredangle YPC = \frac{\pi}{2} - \measuredangle PCA = \frac{\pi}{2} - \measuredangle PH'A = \measuredangle CEH' = \measuredangle HEC$ and hence $l_P \parallel HE$. This means that $M \in l_P$ and that l_P is the P-midline of $\triangle PEH$. Therefore, it passes through the midpoint of PH. We can finish with $\mathbb{H}(H, \frac{1}{2})$ as before. $\square$

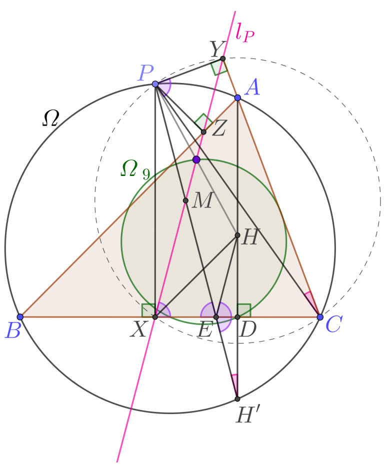

Figure 3: Another proof that Simson line bisects PH

Theorem 5. *Let P and P' be antipodes of Ω. Then P'^*, the isogonal conjugate of P', is the ideal point of l_P.*

Proof. With the same notation as before, because $AK' \parallel l_P$ and because $PYAZ$ is cyclic, $\measuredangle BAK' = \measuredangle AZY = \measuredangle APY$. Further, because $AP \perp AP'$ and $PY \perp AC$, we get $\measuredangle APY = \frac{\pi}{2} - \measuredangle YAP = \measuredangle P'AC$. Therefore, finally, $\measuredangle BAK' = \measuredangle P'AC$, which means that the isogonal conjugate of P' lies on AK' and hence on l_P. □

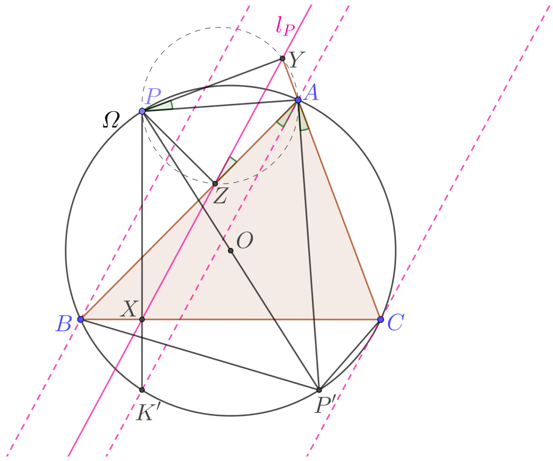

Figure 4: $P'^* \in l_P$

Theorem 6. *If $P, Q \in \Omega$, the $\measuredangle(l_P, l_Q)$ is negative of the angle subtended by arc PQ in Ω.*

Proof. Let perpendiculars from P and Q to BC meet Ω again in P_1 and Q_1 other than P and Q respectively. Then PP_1QQ_1 (not necessarily in that order) is an isosceles trapezium. Moreover, from the first proof of Theorem 4, we know that $l_P \parallel AP_1$ and $l_Q \parallel AQ_1$. Hence, $\measuredangle(l_P, l_Q) = \measuredangle P_1AQ_1 = \measuredangle P_1PQ_1 = -\measuredangle PP_1Q$. Consequently, $\measuredangle(l_P, l_Q) = -\measuredangle PSQ$ for $S \in \Omega$ as required. □

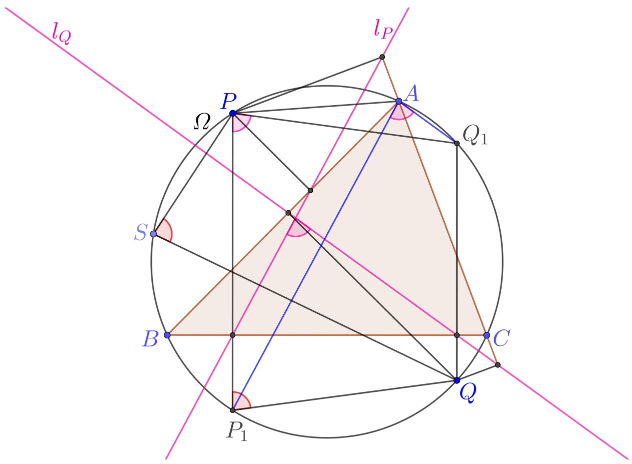

Figure 5: $\measuredangle(l_P, l_Q) = -\measuredangle PSQ$ for $S \in \Omega$. The negative angles are highlighted in a different shade.

Corollary 6.1. *Simson lines of antipodal points are perpendicular.*

Corollary 6.2. *Because isogonal lines of antipodal points are perpendicular, Theorem 5 means that the Simson line of a point is perpendicular to its isogonal line.*

Theorem 7. *Simson lines of antipodal points P and P' of Ω intersect on Ω_9.*

Proof. Let M and M' be the midpoints of PH and PH' respectively. Because $\mathbb{H}(H, \frac{1}{2}) : PP' \mapsto MM'$, M and M' are antipodal on Ω_9. Further, by Corollary 6.1, $l_P \perp l_{P'}$. If $X_P \in l_P \cap l_{P'}$ then $\measuredangle MX_PM' = \frac{\pi}{2}$, because of which $X_P \in \Omega_9$. □

This theorem, as it turns out, links very beautifully with the concept of the asymptotes of rectangular circumhyperbolae, as the following sections will develop.

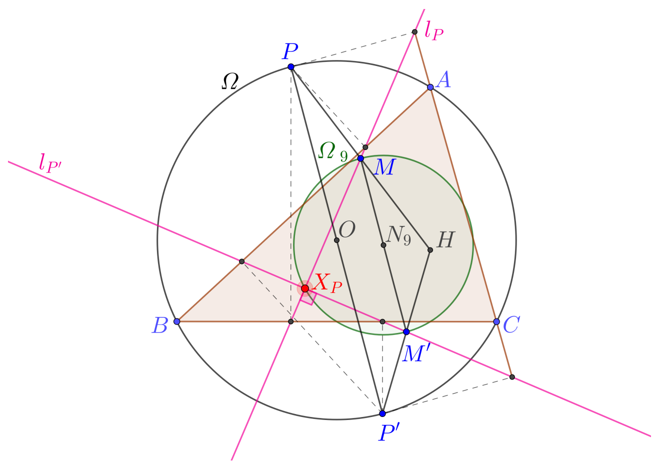

Figure 6: Simson lines of antipodal points meet on Ω_9

3 Two Useful Theorems on Conics

3.1 Brocard's Theorem

Theorem 8 (Brocard's Theorem)**.** *Let $ABCD$ be a quadrilateral inscribed in a conic $\mathcal{C}$. Let $M \in AD \cap BC$, $N \in AB \cap CD$, $P \in AA \cap CC$, $Q \in BB \cap DD$ and $J \in AC \cap BD$. (Here AA means the tangent to $\mathcal{C}$ at A and so on.) Then M, N, P and Q are all collinear on the polar j of J with respect to $\mathcal{C}$.*

Proof. By Pascal's Theorem on $AABCCD$, $P \in MN$. Similarly, by Pascal's Theorem on $ABBCDD$, $Q \in MN$. Moreover, since the J belongs to AC, the polar of P and to BD, the polar of Q, La Hire's Theorem tells us that $PQ \equiv j$ as needed. □

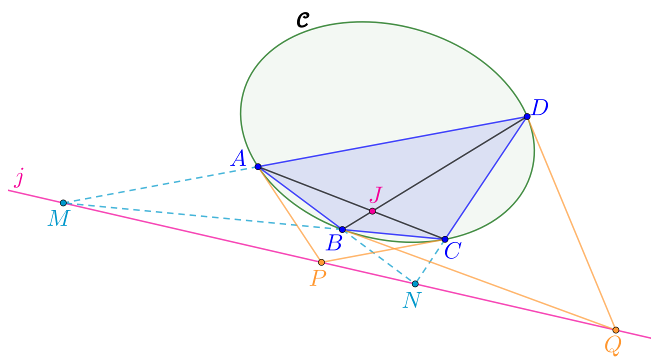

Figure 7: Brocard's Theorem

3.2 8 Points on a Conic

Theorem 9. *For a quadrilateral $ABCD$, assign M, N and J as before, i.e. let $M \in AD \cap BC$, $N \in AB \cap CD$ and $J \in AC \cap BD$. Suppose that quadrilaterals $ABCD$ and $A'B'C'D'$ are assigned the same M, N and J. Then the 8 points $A, B, C, D, A', B', C', D'$ lie on a conic.*

Proof. Consider the projective transformation that maps MN to the ideal line. Then $ABCD$ and $A'B'C'D'$ are mapped to concentric parallelograms with parallel sides which clearly determine the degenerate conic $AC \cup BD$. □

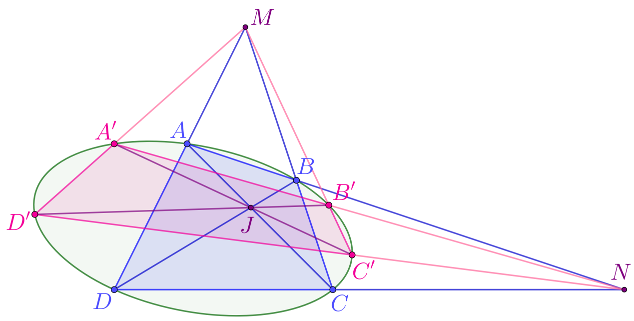

Figure 8: 8 Points on a Conic

4 Onto Rectangular Circumhyperbolae

Since a conic is completely determined by five points, Theorem 3 tells us that a rectangular circumhyperbola of $\triangle ABC$ is characterized completely by the fifth point P it contains. Let $\mathcal{H}(P)$ denote the rectangular circumhyperbola containing P. Further, let Z be the center of $\mathcal{H}(P)$. Then Z is called the Poncelet Point of P with respect to $\triangle ABC$, or in a more symmetric formulation, the Poncelet Point of the quadrilateral $ABCP$.

Theorem 10. *Z lies on the nine-point circle Ω_9 of the orthocentric system $ABCH$.*

Proof. This proof was taken from the online blog linked in [3]. Let D be the fourth intersection of $\mathcal{H}(P)$ with $\Omega \equiv (ABC)$ and let H' be the orthocenter of $\triangle DBC$. By Theorem 3, $H' \in \mathcal{H}(P)$. Moreover, $AHH'D$ is a parallelogram inscribed in a hyperbola $\mathcal{H}(P)$. Applying Brocard's Theorem to $AHH'D$, the center of the parallelogram $AHH'D$ must be the pole of the ideal line l_∞ with respect to $\mathcal{H}(P)$, which is none other than *its* center Z. Hence, Z is the midpoint of HD and once again using $\mathbb{H}(H, \frac{1}{2}) : D \mapsto Z$, we get that $Z \in \Omega_9$. □

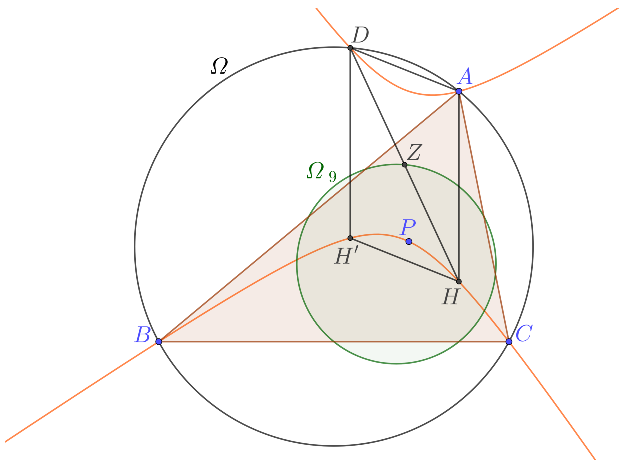

Figure 9: $Z \in \Omega_9$

Corollary 10.1. *Given any four points A, B, C and D in a plane, the nine-point circles of $\triangle ABC$, $\triangle BCD$, $\triangle CDA$ and $\triangle DAB$ concur.*

Proof. Consider the rectangular hyperbola $\mathcal{H}$ that passes through A, B, C and D. Then its center Z, the Poncelet Point of quadrilateral $ABCD$, is the desired point of concurrency. □

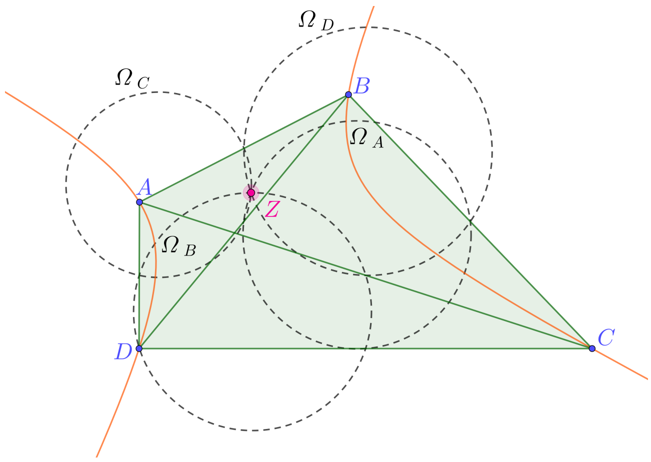

Figure 10: Z as the point of concurrency. Here Ω_A denotes the nine-point circle of $\triangle BCD$ and so on.

Remark. An elementary proof of this result and further reading about the Poncelet Point can be found in [4].

Theorem 11 (Main Theorem)**.** *Let PQ be a diameter of $\Omega \equiv (ABC)$ and let $\mathcal{H}$ denote the rectangular circumhyperbola that is the isogonal conjugate of the line PQ with respect to $\triangle ABC$. Then the asymptotes of $\mathcal{H}$ are the Simson lines l_P and l_Q of P and Q respectively with respect to $\triangle ABC$.*

Proof. As previously, let D be the fourth intersection of $\mathcal{H}$ with Ω and let Z be the center of $\mathcal{H}$. Let O denote the circumcenter of $\triangle ABC$ and let A' denote the antipode of A in Ω. Let $F \in PD \cap BC$ and $E \in l_P \cap BC$. Let P' and Q' be the midpoints of PH and QH respectively. Finally, let the line parallel to PQ through A meet Ω again in G.

We know from Theorem 4 that $P' \in l_P$ and from Theorem 5 that $Q^* \in l_P$. Because Q^* is one of the points at infinity of $\mathcal{H}$, l_P is parallel to one of the asymptotes of $\mathcal{H}$. Hence, to show that it *is* one of the asymptotes, it suffices to show that $Z \in l_P$. The fact that l_Q is the other asymptote follows by symmetry.

We know that BC is the Simson line of A' with respect to $\triangle ABC$. Using Theorem 6, $\measuredangle(BC, l_P) = \measuredangle(l_{A'}, l_P) = -\measuredangle A'QP = \measuredangle PQA'$. But because of the homothety $\mathbb{H}(O, -1) : \triangle A'QP \mapsto \triangle APQ$, $\measuredangle PQA' = \measuredangle QPA$, which in turn equals $\measuredangle GAP$ because of our stipulation that $AG \parallel PQ$. Therefore, $\measuredangle(BC, l_P) = \measuredangle GAP = \measuredangle GAC + \measuredangle CAP$.

Because D is the isogonal conjugate of the ideal point of PQ with respect to $\triangle ABC$, AD and AG are isogonal with respect to $\angle BAC$ implying that $\measuredangle GAC = \measuredangle BAD$. Consequently, $\measuredangle(BC, l_P) = \measuredangle BAD + \measuredangle CAP = \measuredangle BAP + \measuredangle CAD$. However, $\measuredangle BAP = \measuredangle BDP = \measuredangle BDF$ and $\measuredangle CAD = \measuredangle CBD = \measuredangle FBD$. Summing up, $\measuredangle(BC, l_P) = \measuredangle BDF + \measuredangle FBD = \measuredangle BFD = \measuredangle(BC, PD) \implies l_P \parallel PD$.

This means that l_P is the H-midline of $\triangle PDH$ and contains the midpoint of DH, which we know from the proof of the previous theorem to be Z. This concludes the proof. □

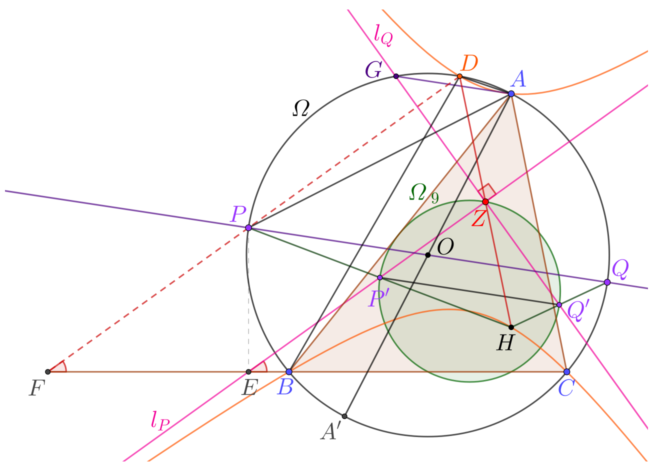

Figure 11: The Turkish Delight!

Remark 11.1. The proof of this theorem that is presented here is due entirely to the author.

5 The Circles Z Belongs To

Let Z be the Poncelet Point of P with respect to $\triangle ABC$. This section develops two cute results taken from [3].

5.1 The Pedal Circle

Theorem 12. *Z lies on the pedal circle of P with respect to $\triangle ABC$.*

Proof. Let D, E, F be the feet of perpendiculars from P to BC, CA, AB respectively. Let K, L, M denote the midpoints of AC, AB, AP respectively.

Then $\measuredangle EZF = \measuredangle EZM + \measuredangle MZF$. Because of Corollary 10.1, $Z \in (MKE) \cap (MLF)$. Hence, $\measuredangle EZM = \measuredangle EKM$ and $\measuredangle MZF = \measuredangle MLF$. Therefore, $\measuredangle EZF = \measuredangle EKM + \measuredangle MLF = \measuredangle AKM + \measuredangle MLA$.

By midlines, it is evident that $\measuredangle AKM = \measuredangle ACP = \measuredangle ECP$. But $ECDP$ is cyclic, so $\measuredangle ECP = \measuredangle EDP$. Consequently, $\measuredangle AKM = \measuredangle EDP$. Similarly, $\measuredangle MLA = \measuredangle PDF$, so that $\measuredangle EZF = \measuredangle AKM + \measuredangle MLA = \measuredangle EDP + \measuredangle PDF = \measuredangle EDF \implies Z \in (DEF)$. □

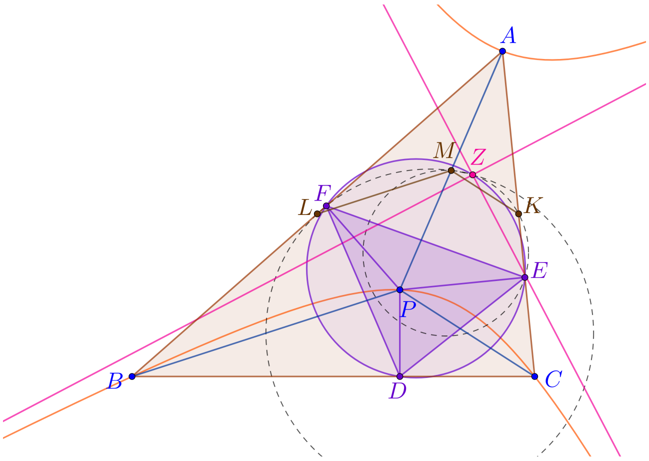

Figure 12: Z lies on the Pedal Circle (DEF)

5.2 The Cevian Circle

Theorem 13. *Z lies on the cevian circle of P with respect to $\triangle ABC$.*

Proof. Let U, V, W be the feet of cevians in $\triangle ABC$. Let J_U, J_V, J_W and I be the corresponding excenters and incenter of $\triangle UVW$. Applying Theorem 9 to quadrilaterals $ABCP$ and $J_U J_V J_W I$ we get that these 8 points lie on a conic. However, I is the orthocenter of $\triangle J_U J_V J_W$, and therefore this conic must be a rectangular hyperbola. Consequently, this conic is nothing but $\mathcal{H}(P)$. To conclude, by Theorem 10, the center Z of this conic lies on the nine-point circle of the orthocentric system $J_U J_V J_W I$, which is nothing but (UVW). □

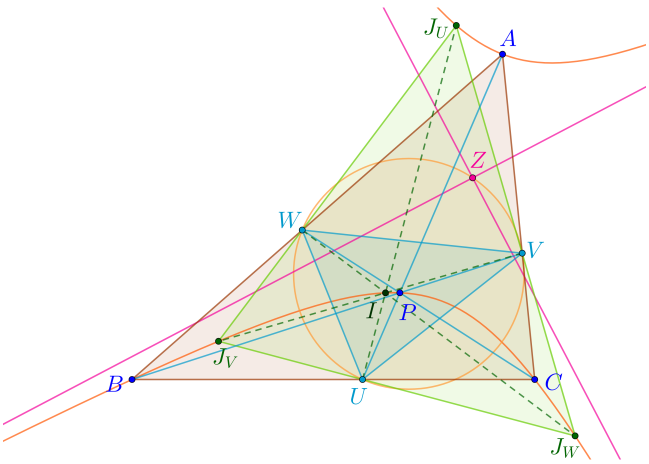

Figure 13: Z lies on the Cevian Circle (UVW)

6 Applications

We end with nice consequences of the theory developed above, which are rather difficult to prove using elementary synthetic geometry.

6.1 The Big Picture

Theorem 14 (Nine Concurrent Circles)**.** *Let A, B, C and D be any four points in a plane. Let Ω_A denote the nine-point circle of $\triangle BCD$, and define Ω_B, Ω_C and Ω_D similarly. Let Γ_A denote the pedal circle of A with respect to $\triangle BCD$, and define Γ_B, Γ_C and Γ_D similarly. Finally, let Λ denote the cevian circle (MNJ) of quadrilateral $ABCD$, where $M \in AD \cap BC$, $N \in AB \cap CD$ and $J \in AC \cap BD$. Then the nine circles $\Omega_A, \Omega_B, \Omega_C, \Omega_D, \Gamma_A, \Gamma_B, \Gamma_C, \Gamma_D$ and Λ concur.*

Proof. This is merely a symmetric formulation of Corollary 10.1 and Theorems 12 and 13. The concurrency point is none other than the Poncelet Point Z of quadrilateral $ABCD$, the center of the rectangular hyperbola through A, B, C and D. □

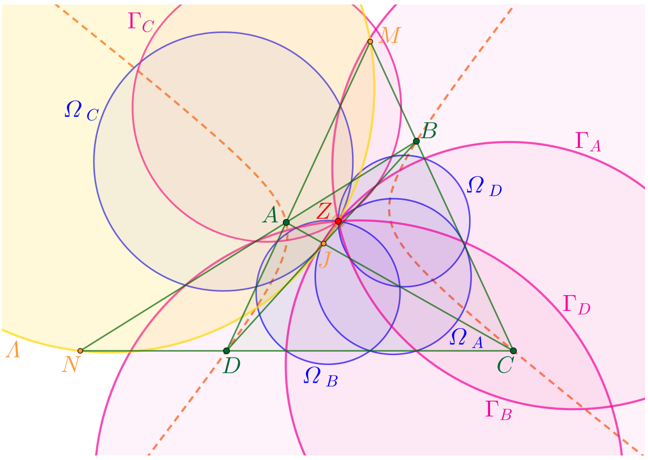

Figure 14: Nine Concurrent Circles

Remark. The other points of intersections include the midpoints of the six sides of $ABCD$ and the feet from one vertex to the segments determined by the other three.

Corollary 14.1 (The Anticenter)**.** *Let $ABCD$ be a cyclic quadrilateral with circumcircle Ω. Let Ω_A denote the nine-point circle of $\triangle BCD$, and define Ω_B, Ω_C and Ω_D similarly. Let l_A denote the Simson Line of A with respect to $\triangle BCD$, and define l_B, l_C and l_D similarly. Finally, let H_A denote the orthocenter of $\triangle BCD$, and define H_B, H_C and H_D similarly. Then $l_A, l_B, l_C, l_D, \Omega_A, \Omega_B, \Omega_C$ and Ω_D concur at the common bisection point of AH_A, BH_B, CH_C and DH_D.*

This point of concurrency is called the anticenter of cyclic quadrilateral $ABCD$.

Proof. In the case when the four points are cyclic, the pedal circles degenerate to Simson Lines. We can use Theorems 4, 10 and 14 to see that the anticenter is none other than the Poncelet Point Z of quadrilateral $ABCD$. □

Corollary 14.2. *With notation as before, quadrilateral $H_AH_BH_CH_D$ is cyclic and homothetic to $ABCD$, the homothety being $\mathbb{H}(Z, -1) : ABCD \mapsto H_AH_BH_CH_D$.*

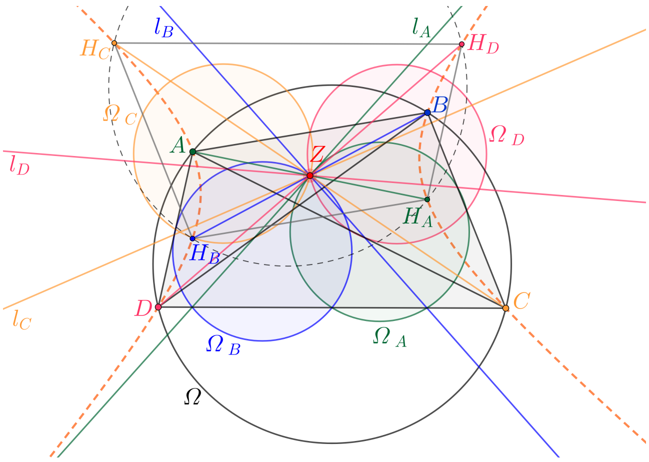

Figure 15: Anticenter

Remark. The standard way to prove the existence of the anticenter is using complex numbers and setting the circumcircle Ω to be the unit circle $\{z : z \in \mathbb{C}, |z| = 1\}$. Then the complex number z denoting the anticenter of points A, B, C and D given by a, b, c and d respectively is given by:

$$z = \frac{a + b + c + d}{2}$$

6.2 Feuerbach's Theorem and the Feuerbach Hyperbola

Theorem 15 (Feuerbach's Theorem)**.** *The nine-point circle Ω_9 of a triangle $\triangle ABC$ is tangent to its incircle ω and three excircles $\omega_A, \omega_B, \omega_C$.*

Proof. This proof was taken from [3]. Let P and Q be isogonal conjugates with respect to $\triangle ABC$, and let O denote the circumcenter of $\triangle ABC$. By the Six Point Circle Theorem, which can be found in [2], they share a common pedal circle; call this pedal circle ω_{PQ}. Then from Theorems 10 and 12, ω_{PQ} meets Ω_9 in the Poncelet Points Z_P and Z_Q of P and Q respectively, with respect to $\triangle ABC$.

Now $\mathcal{H}(P)$ and $\mathcal{H}(Q)$ are distinct lines iff QO and PO are distinct lines, as these are the isogonal conjugates of the hyperbolae. In this case, $Z_P \neq Z_Q$ and $|\omega_{PQ} \cap \Omega_9| = 2$.

If we let the lines PO and QO move closer to each other, the Z_P and Z_Q move closer to each other on the nine-point circle. Consequently if P, O, Q are collinear, then ω_{PQ} and Ω_9 touch. In the particular case when $P \equiv Q$ is the incenter or one of the excenters, we get Feuerbach's Theorem. □

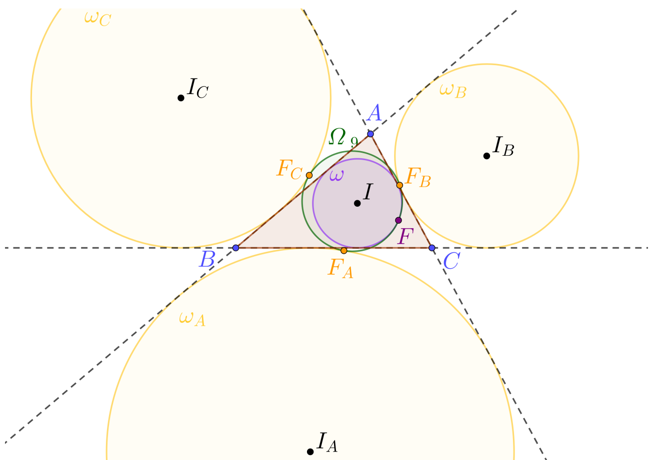

Figure 16: Feuerbach's Theorem

Remark. The point of tangency between ω and Ω_9 is called the Feuerbach Point F of $\triangle ABC$. It is the ETC center X_{11}. The points of tangency with the excircles, denoted by F_A, F_B and F_C respectively form the Feuerbach triangle of $\triangle ABC$.

Theorem 16 (The Feuerbach Hyperbola)**.** *The isogonal conjugate of the line OI of a triangle has its center at F. This hyperbola $\mathcal{H}(I)$, called the Feuerbach Hyperbola of $\triangle ABC$, passes through the Gergonne Point G_E, the Nagel Point N, the Mittenpunkt M and the Schiffler Point S of $\triangle ABC$.*

Proof. The center of the hyperbola $\mathcal{H}(I)$ must lie on the pedal circle ω of I by Theorem 12, and must lie on the nine-point circle Ω_9 of $\triangle ABC$ by Theorem 10. But we know from Theorem 15 that these circles are tangent at F and hence F must be the desired center of $\mathcal{H}(I)$. It is well-known (see [2]) that the isogonal conjugate of the Gergonne Point G_E (X_7) is X_{55}, the in-similicenter of the incircle ω and circumcircle Ω of $\triangle ABC$, and that the isogonal conjugate of the Nagel Point N (X_8) is X_{56}, the ex-similicenter of ω and Ω. Both of these centers of similitude of ω and Ω obviously belong to the line OI and hence their isogonal conjugates belong to $\mathcal{H}(I)$.

The isogonal conjugate of the Mittenpunkt M (X_9) of $\triangle ABC$, called the Isogonal Mittenpunkt, is X_{57}, the homothetic center of the contact and excentral triangles of $\triangle ABC$. This result can be found in [5]. Because this homothetic center takes I to V, the Bevan Point of $\triangle ABC$, it lies on the line $VI \equiv OI$. This shows that the Mittenpunkt lies on $\mathcal{H}(I)$.

Finally, the isogonal conjugate of the Schiffler Point S (X_{21}) of a triangle is the orthocenter of its intouch triangle, labelled X_{65} in the ETC. This can be found in [6]. This point obviously belongs to the Euler Line of the intouch triangle, which is the OI line of the reference triangle $\triangle ABC$. Thus, the Schiffler Point S also lies on $\mathcal{H}(I)$. $\square$

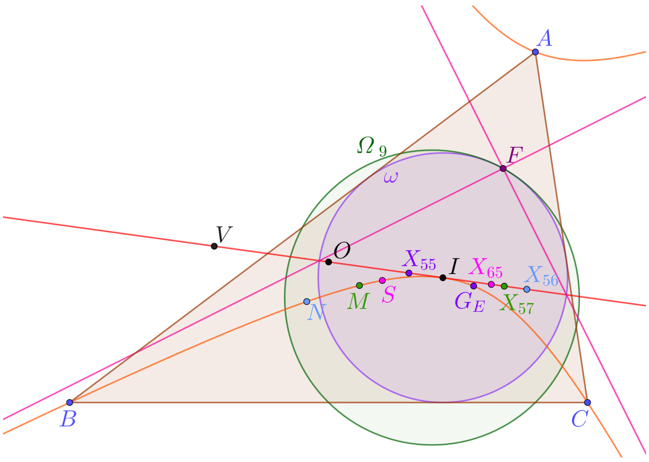

Figure 17: The Feuerbach Hyperbola $\mathcal{H}(I)$

Remark. The line OI is tangent to its isogonal conjugate $\mathcal{H}(I)$. This can be seen by an obvious proof by contradiction.

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