HW1: Linear System Theory (ECE532)

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Problem 1

(a) Assume that output $y(t) = h$ (a constant) in the steady state and $A < 0$. In the steady state, the state variable x does not depend on time anymore, i.e., $\dot{x(t)} = \frac{d}{dt}x(t) = 0$. Therefore, the state-space equations becomes:

$$0 = Ax + Bu \tag{1}$$
$$h = Cx \tag{2}$$

Therefore, $u(t) = -\frac{A}{B}x(t) = -\frac{A}{CB}h$, which proves the required claim.

(b) With the steady state controller $u(t) = -\frac{A}{CB}h$, we can solve the state-space equations by using Laplace transform. Indeed, we substitute $u(t) = -\frac{A}{CB}h$ into the state equation and define $z(t) = x(t) - \frac{h}{C}$, the state equation becomes:

$$\dot{z(t)} = Az(t) \tag{3}$$

So, by applying Laplace transform and inverse Laplace transform which is also shown as follows, we get:

$$\begin{aligned}
\mathcal{L}[\dot{z(t)}] &= \mathcal{L}[Az(t)] \\
sZ(s) - z(0) &= AZ(s) \\
Z(s) &= \frac{z(0)}{s - A} \\
\mathcal{L}^{-1}[Z(s)] &= \mathcal{L}^{-1}[\frac{z(0)}{s - A}] \\
z(t) &= e^{At}z(0) \\
x(t) - \frac{h}{C} &= e^{At}(x(0) - \frac{h}{C}) \\
x(t) &= \frac{h}{C}(1 - e^{At})
\end{aligned}$$

.

Thus,

$$y(t) = Cx(t) = h(1 - e^{At}) \tag{4}$$

for $t \geq 0$.

For $A < 0$,

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} h(1 - e^{At})$$
$$= h$$

(c) For $A > 0$,

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} h(1 - e^{At}) \in \{\infty, -\infty, 0\}$$

depending on whether h is negative, positive or zero, respectively.

(d) Simulation using MATLAB Simulink:

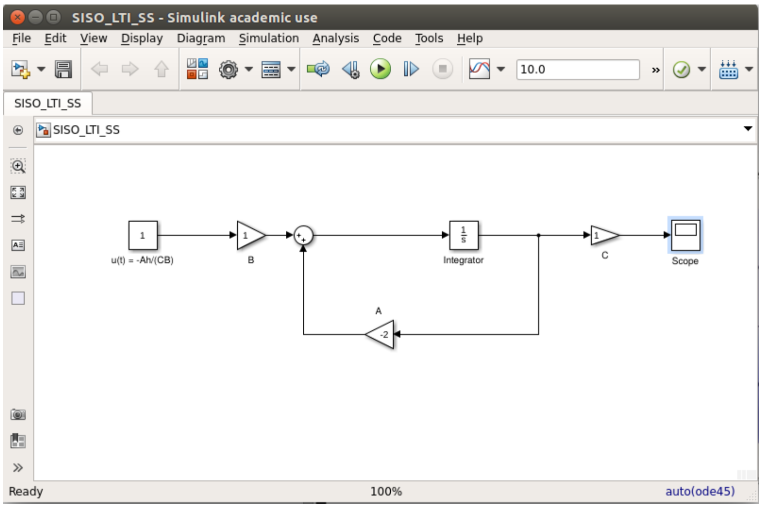

Figure 1: MATLAB Simulink Configuration for $(A, B, C, h) = (-2, 1, 1, 0.5)$

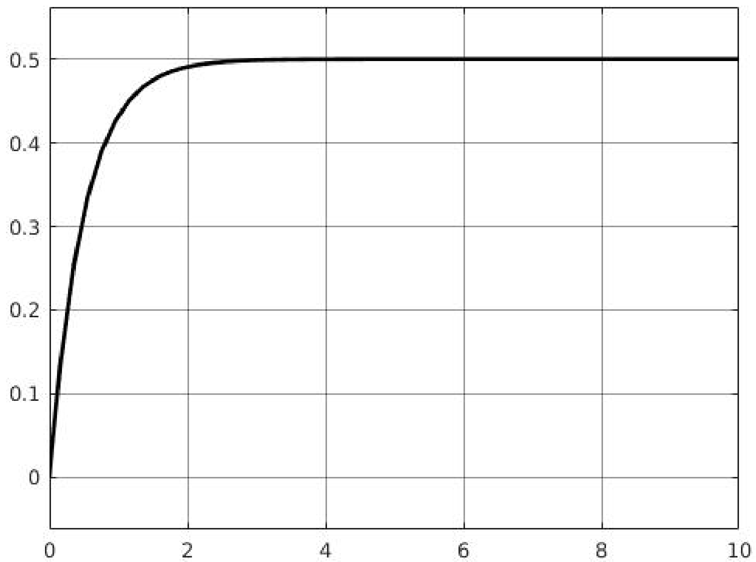

Figure 2: The time response plot for $(A, B, C, h) = (-2, 1, 1, 0.5)$

These plots confirm the correctness the results of the output $y(t)$ in the steady state derived in part (b) and (c).

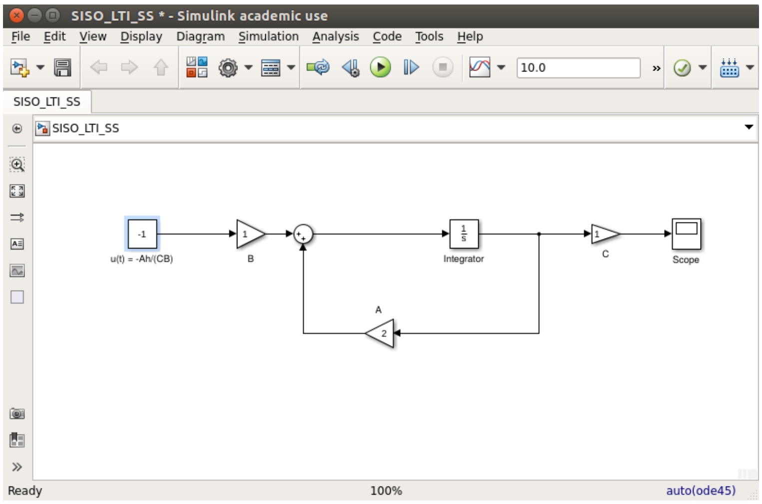

Figure 3: MATLAB Simulink Configuration for $(A, B, C, h) = (2, 1, 1, 0.5)$

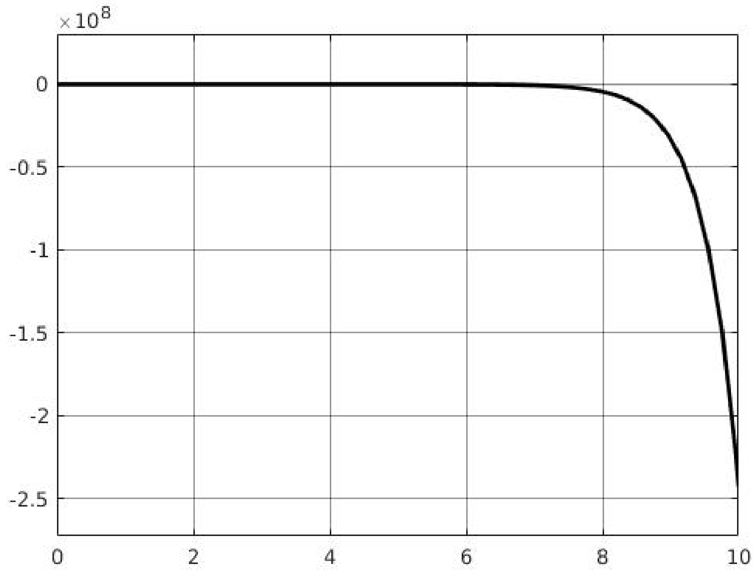

Figure 4: The time response plot for $(A, B, C, h) = (2, 1, 1, 0.5)$

Problem 2

(a) Given the hard disk drive equations, that is,

$$I_1\ddot{\theta}_1 + b(\dot{\theta}_1 - \dot{\theta}_2) + k(\theta_1 - \theta_2) = M_c + M_D \tag{5}$$

$$I_2\ddot{\theta}_2 + b(\dot{\theta}_2 - \dot{\theta}_1) + k(\theta_2 - \theta_1) = 0 \tag{6}$$

we can develop a state equation by choosing $\mathbf{x}(t) = \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix}$ as state variables, $\mathbf{u}(t) = \begin{bmatrix} M_C \\ M_D \end{bmatrix}$ as input variables and $y = \theta_2$ as output variable. For this choice, the state equation for this system is:

$$
\begin{aligned}
\dot{\mathbf{x}} = \begin{bmatrix} \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 & 1 \\ -\frac{k}{I_1} & -\frac{b}{I_1} & \frac{k}{I_1} & \frac{b}{I_1} \\ 0 & 0 & 0 & 1 \\ \frac{k}{I_2} & \frac{b}{I_2} & -\frac{k}{I_2} & -\frac{b}{I_2} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 & 0 \\ \frac{1}{I_1} & \frac{1}{I_1} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{u} \\
&= A\mathbf{x} + Bu \\
y &= \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{x} + 0.\mathbf{u} \\
&= C\mathbf{x}
\end{aligned}
$$

where $A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ -\frac{k}{I_1} & -\frac{b}{I_1} & \frac{k}{I_1} & \frac{b}{I_1} \\ 0 & 0 & 0 & 1 \\ \frac{k}{I_2} & \frac{b}{I_2} & -\frac{k}{I_2} & -\frac{b}{I_2} \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ \frac{1}{I_1} & \frac{1}{I_1} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, and $C = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$.

(b) For $M_D = 0$, $b = 0$, and $\mathbf{y} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$ as output variables, let $u = M_C$ as input variable. The state-space equations become

$$
\begin{aligned}
\dot{\mathbf{x}} &= A\mathbf{x} + Bu \\
\mathbf{y} &= C\mathbf{x}
\end{aligned}
$$

where $A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ -\frac{k}{I_1} & 0 & \frac{k}{I_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{I_2} & 0 & -\frac{k}{I_2} & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ \frac{1}{I_1} \\ 0 \\ 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, and $u = M_C$. Taking Laplace transform both sides of the state-space equations gives the transfer function as follows

$$
\begin{aligned}
\mathbf{H}(s) &= \begin{bmatrix} H_1(s) \\ H_2(s) \end{bmatrix} \\
&= \begin{bmatrix} \frac{Y_1(s)}{U(s)} \\ \frac{Y_2(s)}{U(s)} \end{bmatrix} \\
&= C(sI - A)^{-1}B \\
&= \begin{bmatrix} \frac{(1/I_1)s^2 + k/(I_1 I_2)}{s^4 + (k/I_1)s^2} \\ \\ \frac{k/(I_1 I_2)}{s^4 + (k/I_1)s^2} \end{bmatrix}
\end{aligned}
$$

Problem 3

Assume that the system is operating about the equilibrium point $(\mathbf{x_0}, \mathbf{u_0}) = (\mathbf{0}, \mathbf{0})$ and the variations of $\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t))$ around the equilibrium point is sufficiently small. Then we can write $\mathbf{x}(t) = \mathbf{x_0} + \delta\mathbf{x}(t)$ and $\mathbf{u}(t) = \mathbf{u_0} + \delta\mathbf{u}(t)$.

Recall the vector equation $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t))$, each equation of which $\dot{x}_i(t) = f_i(\mathbf{x}(t), \mathbf{u}(t))$ can be expanded using Taylor series expansion as

$$
\frac{d}{dt}(x_{0i} + \delta x_i) = f_i(\mathbf{x_0} + \delta\mathbf{x}(t), \mathbf{u_0} + \delta\mathbf{u}(t)) \tag{7}
$$

$$
\approx f_i(\mathbf{x_0}, \mathbf{u_0}) + \left.\frac{\partial f_i}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x_0}} \delta\mathbf{x} + \left.\frac{\partial f_i}{\partial \mathbf{u}}\right|_{\mathbf{u}=\mathbf{u_0}} \delta\mathbf{u} \tag{8}
$$

The variations should be small enough for this approximation to hold. Since $\frac{d}{dt}x_{0i} = f_i(\mathbf{x_0}, \mathbf{u_0})$, we thus have

$$
\frac{d}{dt}\delta x_i \approx \left.\frac{\partial f_i}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x_0}} \delta\mathbf{x} + \left.\frac{\partial f_i}{\partial \mathbf{u}}\right|_{\mathbf{u}=\mathbf{u_0}} \delta\mathbf{u} \tag{9}
$$

Combining all n state equations noting that we replace "$\approx$" by "$=$" in (9), gives

$$\frac{d}{dt}\delta\mathbf{x} = \begin{bmatrix} \frac{\partial f_1}{\partial \mathbf{x}}\Big|_{\mathbf{x}=\mathbf{x_0}} \\ \frac{\partial f_2}{\partial \mathbf{x}}\Big|_{\mathbf{x}=\mathbf{x_0}} \\ \vdots \\ \frac{\partial f_n}{\partial \mathbf{x}}\Big|_{\mathbf{x}=\mathbf{x_0}} \end{bmatrix} \delta\mathbf{x} + \begin{bmatrix} \frac{\partial f_1}{\partial \mathbf{u}}\Big|_{\mathbf{u}=\mathbf{u_0}} \\ \frac{\partial f_2}{\partial \mathbf{u}}\Big|_{\mathbf{u}=\mathbf{u_0}} \\ \vdots \\ \frac{\partial f_n}{\partial \mathbf{u}}\Big|_{\mathbf{u}=\mathbf{u_0}} \end{bmatrix} \delta\mathbf{u} \tag{10}$$

$$= A\delta\mathbf{x} + B\delta\mathbf{u} \tag{11}$$

where $A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ & & \vdots & \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}\Bigg|_{\mathbf{x}=\mathbf{x_0}}$ and $B = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_1}{\partial u_n} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \cdots & \frac{\partial f_2}{\partial u_n} \\ & & \vdots & \\ \frac{\partial f_n}{\partial u_1} & \frac{\partial f_n}{\partial u_2} & \cdots & \frac{\partial f_n}{\partial u_n} \end{bmatrix}\Bigg|_{\mathbf{u}=\mathbf{u_0}}$.

Since $\mathbf{x}(t) = \mathbf{x_0} + \delta\mathbf{x}(\mathbf{t}) = \delta\mathbf{x}(\mathbf{t})$ and $\mathbf{u}(t) = \mathbf{u_0} + \delta\mathbf{u}(\mathbf{t}) = \delta\mathbf{u}(\mathbf{t})$, (11) becomes

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t)$$

Problem 4

(a) Choosing $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} r \\ \dot{r} \\ \theta \\ \dot{\theta} \end{bmatrix}$ as state variables, $\mathbf{y} = \begin{bmatrix} r \\ \theta \end{bmatrix}$ as output variables, and $\mathbf{u} = \begin{bmatrix} u_r \\ u_\theta \end{bmatrix}$ as input variables gives the nonlinear state space equation as

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{r} \\ \ddot{r} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \mathbf{f}(\mathbf{x}, \mathbf{u}) = \begin{bmatrix} \dot{r} \\ r\dot{\theta}^2 - k/r^2 + u_r \\ \dot{\theta} \\ -2\dot{r}\dot{\theta}/r + u_\theta/r \end{bmatrix} \tag{12}$$

(b) Let $k = r_0^3\omega_0^2$, we check that $\mathbf{x_0} = \begin{bmatrix} r_0 \\ 0 \\ \omega_0 t \\ \omega_0 \end{bmatrix}$ and $\mathbf{u_0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is one solution to the state space equation (12). Indeed, we can easily see that $\dot{\mathbf{x}}_\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \omega_0 \\ 0 \end{bmatrix}$ and $\mathbf{f}(\mathbf{x_0}, \mathbf{u_0}) = \begin{bmatrix} 0 \\ r_0\omega_0^2 - k/r^2 + 0 \\ \omega_0 \\ -2(0)\omega_0/r_0 + 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \omega_0 \\ 0 \end{bmatrix}$. So, $\dot{\mathbf{x}}_\mathbf{0} = \mathbf{f}(\mathbf{x_0}, \mathbf{u_0})$. We now can obtain a linearized system around the point $(\mathbf{x_0}, \mathbf{u_0})$ by using derived equations from Problem 3. That is,

$$\delta\dot{\mathbf{x}} = A\delta\mathbf{x} + B\delta\mathbf{u}$$
$$\delta\mathbf{y} = C\delta\mathbf{x}$$

where

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_4} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_4} \\ & & \vdots & \\ \frac{\partial f_4}{\partial x_1} & \frac{\partial f_4}{\partial x_2} & \cdots & \frac{\partial f_4}{\partial x_4} \end{bmatrix}\Bigg|_{\mathbf{x}=\mathbf{x_0}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 3\omega_0^2 & 0 & 0 & 2r_0\omega_0 \\ 0 & 0 & 0 & 0 \\ 0 & -2\omega_0/r_0 & 0 & 0 \end{bmatrix}$$

$$B = \left.\begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} \\ \frac{\partial f_3}{\partial u_1} & \frac{\partial f_3}{\partial u_2} \\ \frac{\partial f_4}{\partial u_1} & \frac{\partial f_4}{\partial u_2} \end{bmatrix}\right|_{\mathbf{u}=\mathbf{u_0}} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1/r_0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Problem 5

The system in Figure (a) is linear and the system in Figure (b) and (c) are non linear. In Figure (a), $y(t) = f(x(t)) = kx(t)$ for some non-zero k which satisfies additivity and homogeneity properties for a linear system. In Figure (b), $y(t) = f(x(t)) = kx(t) + y_0$ does not satisfy the additivity condition, that is, $f(x_1(t) + x_2(t)) = k(x_1(t) + x_2(t)) + y_0 \neq f(x_1(t)) + f(x_2(t)) = kx_1(t) + kx_2(t) + 2y_0$. In Figure (c), the graph is a nonlinear curve.

In Figure (b), the system with output $\bar{y}(t) = y(t) - y_0 = g(u(t)) = ku(t)$ is linear.

Problem 6

Let $f : u(t) \rightarrow y(t)$ be the transfer function in the time domain and denote indicator operator $1(.)$ whose value is 1 if its argument is true; otherwise, its value is zero.

(a) **Linearity**

- Additivity

$$\begin{aligned} f(u_1(t) + u_2(t)) &= 1(t \leq \alpha)(u_1(t) + u_2(t)) \\ &= 1(t \leq \alpha)u_1(t) + 1(t \leq \alpha)u_2(t) \\ &= f(u_1(t)) + f(u_2(t)) \end{aligned}$$

  for any inputs $u_1(t)$ and $u_2(t)$.

- Homogeneity

$$\begin{aligned} f(ku(t)) &= 1(t \leq \alpha)ku(t) \\ &= k1(t \leq \alpha)u(t) \\ &= kf(u(t)) \end{aligned}$$

  for any constant k and input $u(t)$.

Therefore, the system is **linear**.

(b) **Time-Invariance**

Consider input $u(t) = 1$, $0 < T < \alpha$, and $y(t) = f(u(t)) = 1(t \leq \alpha)$. We thus have $y(t - T) = 1(t - T \leq \alpha) = 1(t \leq \alpha + T)$. In the other hand, $f(u(t - T)) = f(1) = 1(t \leq \alpha))$. Since $f(u(t - T)) \neq y(t - T)$, the system is **time-variant**.

(c) **Causality**

The output does not depend on future inputs, so the system is **causal**.

Problem 7

Consider the following network

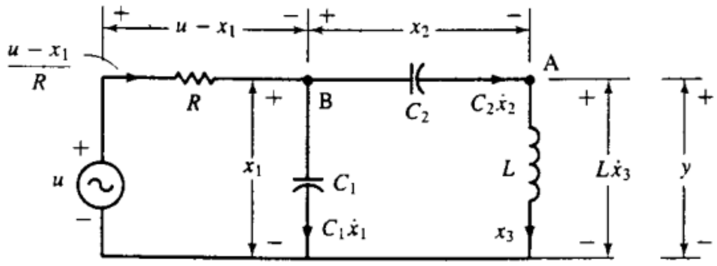

Figure 5: The circuit network

Applying Kirchhoff's current law at node A yields $C_2\dot{x}_2 = x_3$, at node B yields $\frac{u-x_1}{R} = C_1\dot{x}_1 + C_2\dot{x}_2 = C_1\dot{x}_1 + x_3$. We thus have

$$\dot{x}_1 = x_1\frac{-1}{RC_1} + x_3\frac{-1}{C_1} + \frac{u}{RC_1}$$
$$\dot{x}_2 = x_3\frac{1}{C_2}$$

Applying Kirchhoff's voltage law to the right-hand-side loop yields $x_1 - x_2 = L\dot{x}_3$, or

$$y = Lx_3 = x_1 - x_2$$

Choosing $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ as state variables, u as input variable, and y as output variable gives the state space equations for the system

$$\dot{\mathbf{x}} = \begin{bmatrix} -1/RC_1 & 0 & -1/C_1 \\ 0 & 0 & 1/C_2 \\ 1/L & -1/L & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1/RC_1 \\ 0 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}\mathbf{x} + 0.u$$

Assume zero initial state values and take Laplace transform both sides of the state space equations, we have

$$s\mathbf{X}(s) = A\mathbf{X}(s) + BU(s)$$
$$Y(s) = C\mathbf{X}(s)$$

Therefore, the transfer function is

$$\begin{aligned} H(s) &= \frac{Y(s)}{U(s)} \\ &= \frac{C\mathbf{X}(s)}{U(s)} \\ &= C(sI - A)^{-1}B \\ &= \frac{-\frac{1}{RC_1}s^2}{s^3 + \frac{1}{RC_1}s^2 + \frac{1}{L}(\frac{1}{C_1} + \frac{1}{C_2})s + \frac{1}{C_1C_2LR}} \end{aligned}$$

Problem 8

Consider the discrete-time system represented by the difference equation

$$y(k+3) + 2y(k+2) + 3y(k+1) + y(k) = u(k)$$

Choosing $\mathbf{x}(k) = \begin{bmatrix} y(k+2) \\ y(k+1) \\ y(k) \end{bmatrix}$ as state variables, $u(k)$ as input variable, and y(k) as output variable gives the following state space equations

$$\begin{aligned} \mathbf{x}(k+1) &= \begin{bmatrix} y(k+3) \\ y(k+2) \\ y(k+1) \end{bmatrix} \\ &= \begin{bmatrix} -2 & -3 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \mathbf{x}(k) \end{aligned}$$

or

$$\begin{aligned} \mathbf{x}(k+1) &= A\mathbf{x}(k) + Bu(k) \\ y(k) &= C\mathbf{x}(k) + Du(k) \end{aligned}$$

where $A = \begin{bmatrix} -2 & -3 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$, and $D = 0$.

The transfer function can be obtained by directly applying Z-transform to both sides of the difference equation

$$Y(z)z^3 + 2Y(z)z^2 + 3Y(z)z + Y(z) = U(z)$$

So, the transfer function is

$$\begin{aligned} H(z) &= \frac{Y(z)}{U(z)} \\ &= \frac{1}{z^3 + 2z^2 + 3z + 1} \end{aligned}$$

Problem 9

(a) Consider the transfer function

$$\hat{g}(s) = \frac{Y(s)}{U(s)} = \frac{k\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

Taking inverse Laplace transform both sides of the transfer function gives

$$\ddot{y} + 2\xi\omega_n\dot{y} + \omega_n^2 y = k\omega_n^2 u$$

By choosing $\mathbf{x} = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$ as state variables, u as input variable and y as output variable, we have

$$\begin{aligned} \dot{\mathbf{x}} &= \begin{bmatrix} \dot{y} \\ \ddot{y} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\xi\omega_n \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ k\omega_n^2 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x} + 0.u \end{aligned}$$

(b) With the transfer function,

$$\hat{g}(s) = \frac{Y(s)}{U(s)} = \frac{s+a}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

the differential equation becomes

$$\ddot{y} + 2\xi\omega_n\dot{y} + \omega_n^2 y = \dot{u} + au$$

Now, choose $\mathbf{x} = \begin{bmatrix} y \\ \dot{y} \\ u \end{bmatrix}$ as state variables, $\mathbf{u} = \begin{bmatrix} u \\ \dot{u} \end{bmatrix}$ as input variable, and y as output variables. We thus have

$$\begin{aligned}
\dot{\mathbf{x}} &= \begin{bmatrix} \dot{y} \\ \ddot{y} \\ \dot{u} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 0 \\ -\omega_n^2 & -2\xi\omega_n & a \\ 0 & 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{u} \\
y &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{x} + 0.\mathbf{u}
\end{aligned}$$

Problem 10

First, choose $\mathbf{x} = \begin{bmatrix} y_1 \\ \dot{y_1} \\ y_2 \end{bmatrix}$ as state variables, $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ as input variables, and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ as output variables. The state-space equation of the system is

$$\begin{aligned}
\dot{\mathbf{x}} &= \begin{bmatrix} \dot{y_1} \\ \ddot{y_1} \\ \dot{y_2} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 0 \\ -k_2 & -k_1 & 0 \\ 0 & -k_5 & -k_4 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 & 0 \\ 1 & k_3 \\ k_6 & 0 \end{bmatrix} \mathbf{u} \\
y &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x} + 0.\mathbf{u}
\end{aligned}$$