Homework 3
Mathematical Methods I: Fall 2017

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1. **Problem 1 - Elastic Rods:**

   (a) Show that The elastic energy in a bent beam is is

   $$U[y] = \int_0^L \frac{1}{2} YI(y'')^2 dz$$

   given that the elastic energy per unit length of a bent steel rod is given by $\frac{1}{2}\frac{YI}{R^2}$ where R is the radius of curvature due to bending.

   Proof. From vector calculus we know that the length of the radius of curvature vector for a curve $y(x)$ is given by

   $$|\vec{R}| = \frac{(1+y'^2)^{\frac{3}{2}}}{y''} \approx \frac{1}{y''}$$

   where we approximate y'^2 to be small enough to ignore in this case. Then we have

   $$U[y] = \int_0^L \frac{1}{2}\frac{YI}{R^2} dz = \int_0^L \frac{1}{2}\frac{YI}{(}y'')^2 l$$

   □

   (b) Show that if there is a load of mass M on top of the rod, the energy can be approximated by

   $$U[y] = \int_0^L \Big(\frac{YI}{2}(y'')^2 - \frac{Mg}{2}(y')^2\Big) dz$$

   Proof. Gravitational potential energy is clearly going to be

   $$U_g = MgL_z$$

   where L_z is the height of the load after the rod bends. We can calculate this by:

   $$L_z = \int dz$$

   along the curve of the rod. We know that $dl = \sqrt{(}1+y'^2)dz$, so we plug this into

   $$\int dz = \int \frac{dl}{\sqrt{1+y'^2}}$$

   But now we assume that the deflection is very small so we get that $dl \approx dz$ and expand the denominator:

   $$U_g = \int_0^L Mgdz \approx \int_0^L Mg(1 - \frac{1}{2}y'^2)dz$$

   Thus the total energy functional is of the desired form (excluding the constant term MgL). □

(c) Show that the column is unstable to buckling and collapses when $Mg \geq \frac{\pi^2}{L^2}YI$.

Proof. Plugging in the ansatz for the solutions:

$$y(z) = \sum_{n+=1}^{\infty} a_n sin(\frac{n\pi z}{L})$$

into the energy functional:

$$U = \sum_{n=1}^{\infty} a_n^2 \int_0^L \frac{YI}{2}\Big(\frac{n\pi}{L}\Big)^4 sin^2\Big(\frac{n\pi z}{L}\Big) - \frac{Mg}{2}\Big(\frac{n\pi}{L}\Big)^2 cos^2\Big(\frac{n\pi z}{L}\Big)dz$$

Doing the trig integrals out, we get factors of $\frac{1}{2}$ out, and the remaining terms depend on n. We want to know when these terms are negative for a given n value, which would tell us that the energy drops down to a negative value. This first happens at $n = 1$ as the coefficient becomes negative when

$$Mg \geq YI\Big(\frac{n\pi}{L}\Big)^2$$

. □

(d) The light cantilever: Find $y(z)$ for $0 < z < L$ assuming that a rod is fixed into a wall with a load of mass M hanging at the end.

Proof. We want to minimize the energy functional

$$U = \int_0^L \Big(\frac{YI}{2}(y'')^2\Big)dz + Mgy(L)$$

But we will be careful not to throw out terms when we integrate by parts.

$$\begin{aligned}
U(y+\delta y) - U(y) &= \int_0^L \frac{YI}{2}\Big((y+\delta y)^2 - (y'')^2\Big)dz + Mg(y(L) + \delta y(L) - y(L)) \\
\delta U &= \int_0^L \frac{YI}{2}(2y''(\delta y)'' + (\delta y'')^2)dz + Mg\delta y(L) \\
\delta U &= \int_0^L \frac{YI}{2}(2y''(\delta y)'' + \mathcal{O}(\delta y)^2)dz + Mg\delta y(L) \\
\delta U &= \int_0^L YI(y''(\delta y)''dz + Mg\delta y(L)
\end{aligned}$$

We new integrate by parts twice:

$$\begin{aligned}
\delta U &= (y''\delta y)|_0^L - \int_0^L y^{(3)}(\delta y)' + Mg\delta y(L) \\
\delta U &= (y''\delta y)|_0^L - (y^3\delta y)|_0^L + \int_0^L y^{(4)}\delta y dz + Mg\delta y(L) = 0
\end{aligned}$$

This is true for any $\delta y(L)$, knowing that $\delta y(0) = 0$. If we set all these terms to zero, and factor out the terms dependent on we the differential equation

$$y^{(4)} = 0$$

with the boundary conditions

$$\begin{aligned}
Mg &= YIy^{(3)} \\
y''(L) &= 0 \\
y'(0) = y(0) &= 0
\end{aligned}$$

The most general solution to the differential equation is $y(z) = Az^3 + Bz^2 + Cz + D$ But we know immediately that D is zero. From the other boundary conditions we get:

$$y''(L) = 6AL + 2B = 0$$
$$6A = \frac{Mg}{YI}$$

Thus the solution is

$$y(z) = \frac{Mg}{YI}(\frac{1}{6}z^3 - \frac{L}{2}z^2)$$

And

$$y(L) = -\frac{MgL^2}{3YI}$$

□

2. Lagrange Multipliers

   (a) Find the stationary points of the function

$$f(x, y) = 13x^2 + 8xy + 7y^2$$

subject to $x^2 + y^2 = 1$.

Proof. We first express this as a matrix multiplication:

$$\mathbf{x}^t\mathbf{A}\mathbf{x} = f(x, y)$$

Minimizing this function with the constraint gives us an eigenvalue problem to solve:

$$< \mathbf{x}, \mathbf{A}\mathbf{x} > -\lambda(< \mathbf{x}, \mathbf{x} > -1) = g(\mathbf{x})$$

We then differentiate:

$$\frac{\partial g}{\partial \mathbf{x}} = 2\mathbf{A}\mathbf{x} - 2\lambda x = 0$$

And arrive at

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

. Then we can find the eigenvalues of this matrix using the standard techniques. We find them to be

$$\lambda = 10, 5$$

We can also find normalized eigenvectors and we find them to be

$$e_1 = \frac{1}{\sqrt{5}}(2, 1)$$

and

$$e_2 = \frac{1}{\sqrt{5}}(-1, 2)$$

This gives us our two stationary points. But since we know the constraint is in terms of x^2 and y^2, we actually get 4 stationary points that are

$$(x, y) = \pm e_1, \pm e_2$$

□

3. The Catenary Again:

   (a) From the resulting functional derivative, derive two coupled equations for the catenary, one for $x(s)$ and one for $y(s)$.

Proof. We have to minimize the energy functional

$$U(x,y) = \int_0^L \rho g y(s) ds + \int_0^L (\dot{x}^2 + \dot{y}^2 - 1)\lambda(s) ds$$

where s is the coordinate along the curve. We can use euler-lagrange equations

$$\frac{\partial f}{\partial x} = \frac{d}{ds}(\frac{\partial f}{\partial \dot{x}})$$
$$\frac{\partial f}{\partial y} = \frac{d}{ds}(\frac{\partial f}{\partial \dot{y}}) \Rightarrow$$
$$0 = \dot{\lambda}\dot{x} + \lambda\ddot{x}$$
$$\rho g = 2\dot{\lambda}\dot{y} + 2\lambda\ddot{y}$$

Now introduce $\dot{x} = cos\psi, \dot{y} = sin\psi$

$$0 = \dot{\lambda}cos\psi - \lambda sin\psi\dot{\psi}$$
$$\rho g = 2(\dot{\lambda}sin\psi + \lambda cos\psi\dot{\psi})$$

Square these and add them up, we get:

$$(\rho g)^2 = 4(\dot{\lambda}^2 + \lambda^2\dot{\psi}^2)$$

From looking at a section of chain, we can deduce that

$$T(s+ds)_y - T(s)_y = \rho g ds$$

and

$$T(s)_x = T(s+ds)_x$$

If we expand $T(s+ds)_y = T(s)_y + \dot{T}_y ds$ and plug in, we find that $\dot{T}_y = \rho g$. We can then define $T_x = 2\lambda cos\psi$ and $T_y = 2\lambda sin\psi$ such that $\dot{T}_y^2 + \dot{T}_x^2 = (\rho g)^2$. □

(b) Now find the material density $\rho(s)$ in order for a length of chain $\frac{\pi a}{2}$ to hang in an arc of a circle of radius a.

Proof. If we draw the arc for ψ along the arc of the circle, we can deduce that $\psi = \frac{s}{a}$. We also know that $\dot{T}_x = 0$, thus $\frac{d}{ds}(\lambda cos\psi) = 0, \lambda(s)cos\psi = K$. Thus

$$\rho(s)g = 2\partial_s(\lambda sin\psi(s)) = 2\partial_s(Ktan(\frac{s}{a})) = \frac{2K}{a}sec^2(\frac{s}{a})$$

□