Fundamentals of Signal Enhancement and Array Signal Processing
Solution Manual

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10 Beampattern Design

10.1

Show that the minimization of the LSE criterion yields

$$\mathbf{c}_N = \mathbf{M}_{\mathrm{C}}^{-1} \mathbf{v}_{\mathrm{C}} \left(\jmath \bar{f}_m \right).$$

Solution:
First ,from (10.12) we know:

$$LSE(c_N) = 1 - vc^H(j\bar{f}_m)c_N - c_N^H vc + c_N^H M_C c_N$$

we want to find the optimal solution for LSE :

$$\frac{\partial LSE(c_N)}{\partial c_N} = 0$$

$$\rightarrow \frac{\partial LSE(c_N)}{\partial c_N} = -vc(j\bar{f}_m) - vc(j\bar{f}_m) + 2c_N M_C = 0$$

$$\rightarrow c_N = {M_C}^{-1} vc(j\bar{f}_m)$$

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10.2

Show that the elements of the vector $\mathbf{v}_{\mathrm{C}} \left(\jmath \bar{f}_m \right)$ are

$$\left[\mathbf{v}_{\mathrm{C}} \left(\jmath \bar{f}_m \right) \right]_{n+1} = \jmath^n J_n \left(\bar{f}_m \right),$$

where $J_n(z)$ is the Bessel function of the first kind.
Solution:
we can write the vector vc as:

$$vc(j\bar{f}_m) = \frac{1}{\pi} \int_0^{\pi} e^{j\bar{f}_m \cos\theta} P_c(\cos\theta) d\theta$$

$$\rightarrow vc(j\bar{f}_m)_{n+1} = \frac{1}{\pi} \int_0^{\pi} e^{j\bar{f}_m \cos\theta} \cos(n\theta) d\theta$$

let's define:

$$J_n(z) \triangleq \frac{-j^{-n}}{\pi} \int_0^{\pi} e^{jz\cos\theta} \cos(n\theta) d\theta$$

so we can get:

$$vc(j\bar{f}_m)_{n+1} = j^n \cdot J_n(\bar{f}_m)$$

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10.3

Show that the elements of the matrix $\mathbf{M}_\mathrm{C}$ are

$$[\mathbf{M}_\mathrm{C}]_{i+1,j+1} = \frac{1}{\pi}\int_0^\pi \cos(i\theta)\cos(j\theta)\,d\theta.$$

Solution:
The matrix M_c defined as following:

$$M_C = \frac{1}{\pi}\int_0^\pi P_c(\cos\theta)P_c^T(\cos\theta)d\theta$$

using P_c deifinition:

$$\left[P_c(\cos\theta)P_c^T(\cos\theta)\right]_{i+1,j+1} = \cos(i\theta)\cos(j\theta)$$

$$\rightarrow [M_C]_{i+1,j+1} = \frac{1}{\pi}\int_0^\pi \cos(i\theta)\cos(j\theta)d\theta$$

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10.4

Prove the Jacobi-Anger expansion, i.e.,

$$e^{j\bar{f}_m\cos\theta} = \sum_{n=0}^{\infty} \jmath_n J_n\left(\bar{f}_m\right)\cos(n\theta),$$

where

$$\jmath_n = \begin{cases} 1, & n=0 \\ 2j^n, & n=1,2,\ldots,N \end{cases}.$$

Solution:
from 10.11 we know:

$$e^{jf_m\cos\theta} = \lim_{N\to\infty}\sum_{n=0}^{N} c_N\cos(n\theta)$$

where,

$$c_N = [c_0 \quad c_1 \quad \cdots \quad c_N]^T$$
$$c_N = {M_C}^{-1}vc(j\bar{f}_m)$$

from problem 10.2:

$$vc(j\bar{f}_m)_{n+1} = j^n J_n(\bar{f}_m)$$

from problem 10.3:

$$M_C = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \frac{1}{2} & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & \frac{1}{2} \end{pmatrix} \rightarrow {M^{-1}}_C = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & 2 \end{pmatrix}$$

so we get:

$$c_N = \begin{cases} J_0(f_m) & n=0 \\ 2j^nJ_n(f_m) & n\geq 1 \end{cases}$$

subtituting all above:

$$e^{jf_m\cos\theta} = J_0(f_m) + \sum_{n=1}^{\infty} 2j^nJ_n(f_m)\cos(n\theta) = \sum_{n=0}^{\infty} j_n\cdot J_n(f_m)\cos(n\theta)$$

where,

$$j_n = \begin{cases} 1 & n=0 \\ 2j^n & n\geq 1 \end{cases}$$

■

10.6

Show that with the nonrobust filter, $\mathbf{h}_{\mathrm{NR}}(f)$, the first-order beampattern is given by

$$\mathcal{B}_1\left[\mathbf{h}(f), \cos\theta\right] = H_1(f) + J_0\left(\overline{f}_2\right) H_2(f) + 2\jmath J_1\left(\overline{f}_2\right) H_2(f)\cos\theta.$$

Solution:
let's use 10.20 with M=2:

$$B[h(f), \cos\theta] = \sum_{n=0}^{\infty} \cos(n\theta)[\sum_{m=1}^{\infty} j_n J_n(f_m) H_m] =$$

$$= \sum_{n=0}^{\infty} \cos(n\theta)[j_n J_n(f_1) H_1 + j_n J_n(f_2) H_2] =$$

$$= J_0(f_1) H_1(f) + J_0(f_2) H_2 + \sum_{n=1}^{\infty} \cos(n\theta) 2j^n [J_n(f_1) H_1 + J_n(f_2) H_2]$$

We know that :

$$J_0(f_1) = 1$$
$$J_n(f_n) = 0$$

substitute:

$$B[h(f), \cos\theta] = H_1(f) + J_0(f_2) H_2 + \cos(\theta) 2j J_1(f_2) H_2$$

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10.8

Show that by minimizing $J_{\mathrm{FI}}\left[\mathbf{h}(f)\right]$ subject to $\overline{\mathbf{B}}_N(f)\mathbf{h}(f) = \mathbf{b}_N$ and $\mathbf{h}^H(f)\mathbf{h}(f) = \delta_\epsilon$, we obtain the filter:

$$\mathbf{h}_{\mathrm{FI},\epsilon}(f) = \mathbf{\Gamma}_{\mathrm{C},\epsilon}^{-1}(f)\overline{\mathbf{B}}_N^H(f)\left[\overline{\mathbf{B}}_N(f)\mathbf{\Gamma}_{\mathrm{C},\epsilon}^{-1}(f)\overline{\mathbf{B}}_N^H(f)\right]^{-1}\mathbf{b}_N,$$

where $\mathbf{\Gamma}_{\mathrm{C},\epsilon}(f) = \mathbf{\Gamma}_{\mathrm{C}}(f) + \epsilon\mathbf{I}_M$.
Solution:
in order to find corresponding filter we will solve the following minimization:

$$\min \quad h^H(f)\Gamma_c(f)h(f) \quad subject \quad to \quad B_N(f)h(f) = b_n \quad and \quad h^H(f)h(f) = \delta_c$$

using Lagrange multiplier we defined the next function:

$$L(h, \lambda, \varepsilon) = f(h) + \lambda g(h) + \varepsilon k(h)$$

where λ and ϵ is a 1xM vector and :

$$f(h) = h^H(f)\Gamma_c(f)h(f)$$
$$g(h) = B_N(f)h(f) - b_n$$
$$k(h) = h^H(f)h(f) - \delta_c$$

now, finding the min of L:

$$\frac{\partial L(h, \lambda, \varepsilon)}{\partial h} = 0 = 2h(f)\Gamma_c(f) + B_N^H(f)\lambda + 2h(f)\varepsilon$$
$$\rightarrow 2(\Gamma_c(f) + \varepsilon I)h(f) = -B_N^H(f)\lambda$$
$$\rightarrow h(f) = -\frac{1}{2}(\Gamma_c(f) + \varepsilon I)^{-1} B_N^H \lambda$$
$$\frac{\partial L(h, \lambda, \varepsilon)}{\partial \lambda} = 0 \rightarrow B_N(f)h(f) = b_n$$
$$B_N(f)h(f) = -\frac{1}{2} B_N(f)(\Gamma_c(f) + \varepsilon I)^{-1} B_N^H \lambda = b_n$$

$$\to \lambda = -2\Big(B_N(f)(\Gamma_c(f)+\varepsilon I)^{-1}B_N^H\Big)^{-1}b_n$$

$$\to h(f) = (\Gamma_c(f)+\varepsilon I)^{-1}B_N^H\Big(B_N(f)(\Gamma_c(f)+\varepsilon I)^{-1}B_N^H\Big)^{-1}b_n =$$

$$= {\Gamma_{c,\varepsilon}}^{-1}(f)B_N^H\big(B_N(f){\Gamma_{c,\varepsilon}}^{-1}(f)B_N^H\big)^{-1}b_n$$

where,

$${\Gamma_{c,\varepsilon}}^{-1}(f) \triangleq \Gamma_c(f)+\varepsilon I$$

■

10.9

Show that the LSE between the array beampattern and the desired directivity pattern can be written as

$$\text{LSE}\left[\mathbf{h}(f)\right] = \mathbf{h}^H(f)\mathbf{\Gamma}_\text{C}(f)\mathbf{h}(f) - \mathbf{h}^H(f)\mathbf{\Gamma}_{\mathbf{dp}_\text{C}}(f)\mathbf{b}_N - \\ \mathbf{b}_N^T\mathbf{\Gamma}_{\mathbf{dp}_\text{C}}^H(f)\mathbf{h}(f) + \mathbf{b}_N^T\mathbf{M}_\text{C}\mathbf{b}_N.$$

Solution:
let's remember the definition of LSE:

$$LSE[h(f)] = \frac{1}{\pi}\int_0^\pi |\varepsilon[h(f),\cos\theta]|^2 d\theta$$

where,

$$|\varepsilon[h(f),\cos\theta]|^2 = \left|d^H(f,\cos\theta)h(f) - {P_c}^T(\cos\theta)b_N\right|^2 =$$

$$= \left(h^H(f)d(f,\cos\theta) - b_N^H P_c(\cos\theta)\right)\left(d^H(f,\cos\theta)h(f) - {P_c}^T(\cos\theta)b_N\right) =$$

$$= h^H(f)d(f,\cos\theta)d^H(f,\cos\theta)h(f) - h^H(f)d(f,\cos\theta){P_c}^T(\cos\theta)b_N - b_N^H P_c(\cos\theta)d^H(f,\cos\theta)h(f) + \\ + b_N^H P_c(\cos\theta){P_c}^T(\cos\theta)b_N$$

substituting:

$$LSE[h(f)] = \frac{1}{\pi}\int_0^\pi |\varepsilon[h(f),\cos\theta]|^2 d\theta =$$

$$= \frac{1}{\pi}\int_0^\pi h^H(f)d(f,\cos\theta)d^H(f,\cos\theta)h(f)d\theta - \frac{1}{\pi}\int_0^\pi h^H(f)d(f,\cos\theta){P_c}^T(\cos\theta)b_N d\theta - \\ - \frac{1}{\pi}\int_0^\pi b_N^H P_c(\cos\theta)d^H(f,\cos\theta)h(f)d\theta + \frac{1}{\pi}\int_0^\pi b_N^H P_c(\cos\theta){P_c}^T(\cos\theta)b_N d\theta =$$

$$= h^H(f)\left[\frac{1}{\pi}\int_0^\pi d(f,\cos\theta)d^H(f,\cos\theta)d\theta\right]h(f) - h^H(f)\left[\frac{1}{\pi}\int_0^\pi d(f,\cos\theta){P_c}^T(\cos\theta)d\theta\right]b_N - \\ - b_N^H\left[\frac{1}{\pi}\int_0^\pi P_c(\cos\theta)d^H(f,\cos\theta)d\theta\right]h(f) + b_N^H\left[\frac{1}{\pi}\int_0^\pi P_c(\cos\theta){P_c}^T(\cos\theta)d\theta\right]b_N$$

using the following definitions:

$$\Gamma_c(f) \triangleq \frac{1}{\pi}\int_0^\pi d(f,\cos\theta)d^H(f,\cos\theta)d\theta$$

$$\Gamma_{dpc}(f) \triangleq \frac{1}{\pi}\int_0^\pi d(f,\cos\theta){P_c}^T(\cos\theta)d\theta$$

$$M_C \triangleq \frac{1}{\pi}\int_0^\pi P_c(\cos\theta){P_c}^T(\cos\theta)d\theta$$

so,

$$LSE[h(f)] = h^H(f)\Gamma_c(f)h(f) - h^H(f)\Gamma_{dpc}(f)b_N - b_N^H\Gamma_{dpc}^H(f)h(f) + b_N^H M_C b_N$$

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10.10

Show that by minimizing the LSE with a constraint on the coefficients, we obtain the regularized LS filter:

$$\mathbf{h}_{\mathrm{LS},\epsilon}(f) = \mathbf{\Gamma}_{\mathrm{C},\epsilon}^{-1}(f)\mathbf{\Gamma}_{\mathbf{dp}_{\mathrm{C}}}(f)\mathbf{b}_N.$$

Solution:
in order to find the LS filter we will solve the following minimization:

$$\min \quad h^H(f)\Gamma_c(f)h(f) - h^H(f)\Gamma_{dpc}(f)b_N - {b_N}^H(f){\Gamma^H}_{dpc}(f)h(f) + {b_N}^T M_c b_N$$
$$subject \quad to \quad h^H(f)h(f) = \delta_c$$

using Lagrange multiplier we defined the next function:

$$L(h, \lambda, \varepsilon) = f(h) + \varepsilon g(h)$$

where ϵ is a 1xM vector and :

$$f(h) = h^H(f)\Gamma_c(f)h(f)$$
$$g(h) = h^H(f)h(f) - \delta_c$$

now, finding the min of L:

$$\frac{\partial L(h, \varepsilon)}{\partial h} = 0 \rightarrow 2\Gamma_c(f)h(f) - \Gamma_{dpc}(f)b_N - \Gamma_{dpc}(f)b_N + 2\varepsilon h(f)$$
$$\rightarrow 2(\Gamma_c(f) + \varepsilon I)h(f) = 2\Gamma_{dpc}(f)b_N$$
$$\rightarrow h_{LS,e} = (\Gamma_c(f) + \varepsilon I)^{-1}\Gamma_{dpc}(f)b_N = \Gamma_{c,\varepsilon}(f)^{-1}\Gamma_{dpc}(f)b_N$$

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10.11

Show that by minimizing the LSE subject to the distortionless constraint and a constraint on the coefficients, we obtain the regularized CLS filter:

$$\mathbf{h}_{\mathrm{CLS},\epsilon}(f) = \mathbf{h}_{\mathrm{LS},\epsilon}(f) - \frac{1 - \mathbf{d}^H(f,1)\,\mathbf{h}_{\mathrm{LS},\epsilon}(f)}{\mathbf{d}^H(f,1)\,\mathbf{\Gamma}_{\mathrm{C},\epsilon}^{-1}(f)\mathbf{d}(f,1)}\mathbf{\Gamma}_{\mathrm{C},\epsilon}^{-1}(f)\mathbf{d}(f,1).$$

Solution:
in order to find the CLS filter we will solve the following minimization:

$$\min \quad LSE|\varepsilon|^2 \quad subject \quad to \quad h^H(f)d(f,\cos\theta) = 1 \quad and \quad h^H(f)h(f) = \delta_c$$

using Lagrange multiplier we defined the next function:

$$L(h, \lambda, \varepsilon) = f(h) + \lambda g(h) + \varepsilon k(h)$$

where λ and ϵ is a 1xM vector and

$$f(h) = LSE[h(f)]$$
$$g(h) = h^H(f)d(f,\cos\theta) - 1$$
$$k(h) = h^H(f)h(f) - \delta_c$$

now, finding the min of L:

$$\frac{\partial L(h, \lambda, \varepsilon)}{\partial h} = 0 \rightarrow 2\Gamma_c(f)h(f) - \Gamma_{dpc}(f)b_N - \Gamma_{dpc}(f)b_N + 2\varepsilon h(f) + \lambda d(f,\cos\theta)$$
$$\rightarrow 2(\Gamma_c(f) + \varepsilon I)h(f) = 2\Gamma_{dpc}(f)b_N - \lambda d(f,\cos\theta)$$
$$\rightarrow h_{CLS} = (\Gamma_c(f) + \varepsilon I)^{-1}[\Gamma_{dpc}(f)b_N - \frac{1}{2}\lambda d(f,\cos\theta)] = h_{LS,e}(f) - \frac{1}{2}\lambda {\Gamma^{-1}}_{c,e} d(f,\cos\theta)$$
$$\frac{\partial L(h, \lambda, \varepsilon)}{\partial \lambda} = 0 \rightarrow h^H(f)d(f,\cos\theta) = 1 \rightarrow d^H(f,\cos\theta)h(f) = 1$$

$$d^H(f,\cos\theta)h(f) = d^H(f,\cos\theta)h_{LS,e}(f) - \frac{1}{2}\lambda d^H(f,\cos\theta)\Gamma^{-1}{}_{c,e}d(f,\cos\theta) = 1$$
$$\rightarrow \lambda = -2\frac{1 - d^H(f,\cos\theta)h_{LS,e}(f)}{d^H(f,\cos\theta)\Gamma^{-1}{}_{c,e}d(f,\cos\theta)}$$
$$\rightarrow h_{CLS} = h_{LS,e}(f) + \frac{1 - d^H(f,\cos\theta)h_{LS,e}(f)}{d^H(f,\cos\theta)\Gamma^{-1}{}_{c,e}d(f,\cos\theta)}\Gamma^{-1}{}_{c,e}d(f,\cos\theta)$$

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10.12

Show that with the constraint $\overline{\mathbf{B}}_N(f)\mathbf{h}(f) = \mathbf{b}_N$, the error signal between the array beampattern and the desired directivity pattern can be expressed as

$$\mathcal{E}\left[\mathbf{h}(f),\cos\theta\right] = \sum_{i=N+1}^{\infty} \cos(i\theta)\,\overline{\mathbf{b}}_i^T(f)\mathbf{h}(f).$$

Solution:
from 10.55 we know:

$$\varepsilon[h(f),\cos\theta] = \sum_{i=0}^{\infty}\cos(i\theta)\bar{b}_i^T h(f) - \sum_{i=0}^{N}\cos(i\theta)\bar{b}_{N,i} =$$
$$= \sum_{i=N+1}^{\infty}\cos(i\theta)\bar{b}_i^T h(f) + \sum_{i=0}^{N}\cos(i\theta)\bar{b}_i^T h(f) - \sum_{i=0}^{N}\cos(i\theta)\bar{b}_{N,i}$$

using the following constraint:

$$B_N(f)h(f) = b_N$$
$$\rightarrow {b_i}^T h(f) = \bar{b}_{N,i}$$

substituting:

$$\varepsilon[h(f),\cos\theta] = \sum_{i=N+1}^{\infty}\cos(i\theta)\bar{b}_i^T h(f) + \sum_{i=0}^{N}\cos(i\theta)\bar{b}_i^T h(f) - \sum_{i=0}^{N}\cos(i\theta){b_i}^T h(f) =$$
$$= \sum_{i=N+1}^{\infty}\cos(i\theta)\bar{b}_i^T h(f)$$

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10.13

Using the orthogonality property of the Chebyshev polynomials, show that the criterion $J_{\mathrm{FI}}\left[\mathbf{h}(f)\right]$ can be expressed as

$$J_{\mathrm{FI}}\left[\mathbf{h}(f)\right] = \mathrm{LSE}\left[\mathbf{h}(f)\right] + \frac{1}{\pi}\int_0^{\pi}\left|\mathcal{B}\left(\mathbf{b}_N,\cos\theta\right)\right|^2 d\theta,$$

where

$$\mathrm{LSE}\left[\mathbf{h}(f)\right] = \frac{1}{\pi}\int_0^{\pi}\left|\sum_{i=N+1}^{\infty}\cos(i\theta)\,\overline{\mathbf{b}}_i^T(f)\mathbf{h}(f)\right|^2 d\theta.$$

Solution:
first we know that:

$$LSE[h(f)] = \frac{1}{\pi}\int_0^{\pi}|\varepsilon[h(f),\cos\theta]|^2 d\theta = \frac{1}{\pi}\int_0^{\pi}\left|\sum_{i=N+1}^{\infty}\cos(i\theta){b_i}^T h(f)\right|^2 d\theta$$

now, using 10.56 the criterion defined in 10.40 can be expressed as:

$$J_{FI}[h(f)] = \frac{1}{\pi}\int_0^{\pi} |\varepsilon[h(f),\cos\theta] + B[b_N,\cos\theta]|^2 d\theta = \frac{1}{\pi}\int_0^{\pi} \left| \sum_{i=N+1}^{\infty} \cos(i\theta){b_i}^T h(f) + \sum_{i=0}^{N} \cos(i\theta) b_{N,i} \right|^2 d\theta$$

using the orthogonality property:

$$\int_0^{\pi} \cos(i\theta)\cos(j\theta)d\theta = 0 \quad i \neq j$$

and now:

$$J_{FI}[h(f)] = \frac{1}{\pi}\int_0^{\pi} \left| \sum_{i=N+1}^{\infty} \cos(i\theta){b_i}^T h(f) \right|^2 d\theta + \frac{1}{\pi}\int_0^{\pi} \left| \sum_{i=0}^{N} \cos(i\theta) b_{N,i} \right|^2 d\theta =$$

$$= \frac{1}{\pi}\int_0^{\pi} |\varepsilon[h(f),\cos\theta]|^2 d\theta + \frac{1}{\pi}\int_0^{\pi} |B[b_N,\cos\theta]|^2 d\theta$$

$$\rightarrow J_{FI}[h(f)] = LSE[h(f)] + \frac{1}{\pi}\int_0^{\pi} |B[b_N,\cos\theta]|^2 d\theta$$

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10.14

Show that the filters defined in (**??**) preserve the nulls of $\mathbf{h}'(f) = \mathbf{h}_{\mathrm{NR}}(f)$, i.e., if θ_0 is a null of $\mathbf{h}'(f)$, then

$$\mathbf{h}^H(f)\mathbf{d}(f,\cos\theta_0) = \mathbf{g}^H(f)\widetilde{\mathbf{d}}(f,\cos\theta_0) \times 0 = 0,$$

where

$$\widetilde{\mathbf{d}}(f,\cos\theta_0) = \begin{bmatrix} 1 & e^{-\jmath 2\pi f\tau_0\cos\theta_0} & \cdots & e^{-\jmath(M-N-1)2\pi f\tau_0\cos\theta_0} \end{bmatrix}^T.$$

Solution:
we know the form of h is:

$$h(f) = H^{'}(f)g(f) \rightarrow h^H(f) = g^H(f)H^{'H}(f)$$

let's define:

$$\tilde{d}(f,\cos\theta) \triangleq d(f,\cos\theta)H^{'H} = [1 \quad e^{-j2\pi f\tau_0\cos\theta} \quad \cdots \quad e^{-j(M-N-1)2\pi f\tau_0\cos\theta}]^T$$

if θ_0 is a null of $h'(f) = h_{NR}(f)$ so:

$$h^H(f)d(f,\cos\theta) = g^H(f)\tilde{d}(f,\cos\theta_0) \times 0 = 0$$

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10.15

Show that by maximizing the WNG subject to the distortionless constraint, we obtain the MWNG filter:

$$\mathbf{h}_{\mathrm{MWNG}}(f) = \frac{\mathbf{H}'(f)\left[\mathbf{H}'^H(f)\mathbf{H}'(f)\right]^{-1}\widetilde{\mathbf{d}}(f,1)}{\widetilde{\mathbf{d}}^H(f,1)\left[\mathbf{H}'^H(f)\mathbf{H}'(f)\right]^{-1}\widetilde{\mathbf{d}}(f,1)}.$$

Solution:
in order to find the MWNG filter we will solve the following minimization:

$$\min \quad g^H(f)H^{'H}(f)H'(f)g(f) \quad subject \quad to \quad g^H(f)\tilde{d}(f,1) = 1$$

using Lagrange multiplier we defined the next function:

$$L(g,\lambda) = f(g) + \lambda k(g)$$

where λ is a 1xM vector and

$$f(g) = g^H(f)H^{'H}(f)H'(f)g(f)$$
$$k(g) = g^H(f)\tilde{d}(f,1) - 1$$

now, finding the min of L:

$$\frac{\partial L(g,\lambda)}{\partial g} = 0 = 2H^{'H}(f)H^{'}(f)g(f) + \tilde{d}(f,1)\lambda$$
$$\rightarrow g(f) = -\frac{1}{2}\left[H^{'H}(f)H^{'}(f)\right]^{-1}\tilde{d}(f,1)\lambda$$
$$\frac{\partial L(g,\lambda)}{\partial \lambda} = 0 \rightarrow g^H(f)\tilde{d}(f,1) = 1 \rightarrow \tilde{d}^H(f,1)g(f) = 1$$
$$\tilde{d}^H(f,1)g(f) = -\frac{1}{2}\tilde{d}^H(f,1)\left[H^{'H}(f)H^{'}(f)\right]^{-1}\tilde{d}(f,1)\lambda = 1$$
$$\rightarrow \lambda = -\frac{2}{\tilde{d}^H(f,1)[H'^H(f)H'(f)]^{-1}\tilde{d}(f,1)}$$
$$\rightarrow g_{MWNG}(f) = \frac{\left[H^{'H}(f)H^{'}(f)\right]^{-1}\tilde{d}(f,1)}{\tilde{d}^H(f,1)[H'^H(f)H'(f)]^{-1}\tilde{d}(f,1)}$$

use the form of h(f):

$$h(f) = H^{'H}(f)g(f)$$
$$\rightarrow h_{MWNG}(f) = \frac{H^{'H}(f)\left[H^{'H}(f)H^{'}(f)\right]^{-1}\tilde{d}(f,1)}{\tilde{d}^H(f,1)[H'^H(f)H'(f)]^{-1}\tilde{d}(f,1)}$$

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10.16

Show that by minimizing $J_\aleph[\mathbf{g}(f)]$ subject to the distortionless constraint, we obtain the tradeoff filter:

$$\mathbf{g}_{\mathrm{T},\aleph}(f) = \mathbf{g}_{\mathrm{U},\aleph}(f) + \frac{1-\widetilde{\mathbf{d}}^H(f,1)\,\mathbf{g}_{\mathrm{U},\aleph}(f)}{\widetilde{\mathbf{d}}^H(f,1)\,\mathbf{R}_\aleph^{-1}(f)\widetilde{\mathbf{d}}(f,1)}\mathbf{R}_\aleph^{-1}(f)\widetilde{\mathbf{d}}(f,1),$$

where

$$\mathbf{g}_{\mathrm{U},\aleph}(f) = \aleph\mathbf{R}_\aleph^{-1}(f)\mathbf{H}'^H(f)\mathbf{\Gamma}_{\mathbf{dp}_\mathrm{C}}(f)\mathbf{b}_N$$

is the unconstrained filter obtained by minimizing $J_\aleph[\mathbf{g}(f)]$ and

$$\mathbf{R}_\aleph(f) = \aleph\mathbf{R}(f) + (1-\aleph)\mathbf{H}'^H(f)\mathbf{H}'(f).$$

Solution:
in order to find the tradeoff filter we will solve the following minimization:

$$\min\ \aleph LSE[g(f)] + (1-\aleph)\,\mathrm{g}^H(f)H^{'H}(f)H^{'}(f)\mathrm{g}(f)\ \textit{subject to}\ g^H(f)\tilde{d}(f,1) = 1$$

using Lagrange multiplier we defined the next function:

$$L(g,\lambda) = f(g) + \lambda k(g)$$

where λ is a 1xM vector and

$$f(g) = \aleph LSE[g(f)] + (1-\aleph)\,\mathrm{g}^H(f)H^{'H}(f)H^{'}(f)\mathrm{g}(f)$$
$$k(g) = g^H(f)\tilde{d}(f,1) - 1$$

we know:

$$LSE\left[g(f)\right] = \mathrm{g}^{H}(f)H^{'H}(f)\Gamma_{C}(f)H^{'}(f)\mathrm{g}(f) - \mathrm{g}^{H}(f)H^{'H}(f)\Gamma_{dpc}(f)b_{N} - b_{N}^{T}\Gamma_{dpc}^{H}(f)H^{'}(f)\mathrm{g}(f) + b_{N}^{T}M_{c}b_{N}$$

now, finding the min of L:

$$\frac{\partial L(g,\lambda)}{\partial g} = 0 = \aleph\left[2\left(H^{'H}(f)\Gamma_{C}(f)H^{'}(f)\right)g(f) - H^{'H}(f)\Gamma_{dpc}(f)b_{N} - H^{'H}(f)\Gamma_{dpc}(f)b_{N}\right] +$$

$$+2\left(1-\aleph\right)H^{'H}(f)H^{'}(f)\mathrm{g}(f) + \tilde{d}(f,1)\lambda = 0$$

$$\rightarrow \left[2\aleph\left(H^{'H}(f)\Gamma_{C}(f)H^{'}(f)\right) + 2\left(1-\aleph\right)H^{'H}(f)H^{'}(f)\right]g(f) = 2\aleph H^{'H}(f)\Gamma_{dpc}(f)b_{N} - \tilde{d}(f,1)\lambda$$

$$\rightarrow g(f) = \left[\aleph\left(H^{'H}(f)\Gamma_{C}(f)H^{'}(f)\right) + \left(1-\aleph\right)H^{'H}(f)H^{'}(f)\right]^{-1}\left[\aleph H^{'H}(f)\Gamma_{dpc}(f)b_{N} - \frac{1}{2}\tilde{d}(f,1)\lambda\right]$$

we know from 10.76:

$$R(f) = H^{'H}(f)\Gamma_{C}(f)H^{'}(f)$$

$$\rightarrow g(f) = \left[\aleph R(f) + \left(1-\aleph\right)H^{'H}(f)H^{'}(f)\right]^{-1}\left[\aleph H^{'H}(f)\Gamma_{dpc}(f)b_{N} - \frac{1}{2}\tilde{d}(f,1)\lambda\right]$$

let's dfine:

$$R_{\aleph}(f) \triangleq \aleph R(f) + \left(1-\aleph\right)H^{'H}(f)H^{'}(f)$$

$$\rightarrow g(f) = \aleph R_{\aleph}(f)^{-1}H^{'H}(f)\Gamma_{dpc}(f)b_{N} - \frac{1}{2}R_{\aleph}(f)^{-1}\tilde{d}(f,1)\lambda$$

find λ :

$$\frac{\partial L(g,\lambda)}{\partial \lambda} = 0 \rightarrow g^{H}(f)\tilde{d}(f,1) = 1 \rightarrow \tilde{d}^{H}(f,1)g(f) = 1$$

$$\tilde{d}^{H}(f,1)g(f) = \aleph\tilde{d}^{H}(f,1)R_{\aleph}(f)^{-1}H^{'H}(f)\Gamma_{dpc}(f)b_{N} - \frac{1}{2}\tilde{d}^{H}(f,1)R_{\aleph}(f)^{-1}\tilde{d}(f,1)\lambda$$

$$\rightarrow \lambda = -2\frac{1-\aleph\tilde{d}^{H}(f,1)R_{\aleph}(f)^{-1}H^{'H}(f)\Gamma_{dpc}(f)b_{N}}{\tilde{d}^{H}(f,1)R_{\aleph}(f)^{-1}\tilde{d}(f,1)}$$

substituting

$$\rightarrow g_{T,\aleph}(f) = \aleph R_{\aleph}(f)^{-1}H^{'H}(f)\Gamma_{dpc}(f)b_{N} + R_{\aleph}(f)^{-1}\tilde{d}(f,1)\frac{1-\aleph\tilde{d}^{H}(f,1)R_{\aleph}(f)^{-1}H^{'H}(f)\Gamma_{dpc}(f)b_{N}}{\tilde{d}^{H}(f,1)R_{\aleph}(f)^{-1}\tilde{d}(f,1)}$$

define:

$$g_{U,\aleph}(f) \triangleq \aleph R_{\aleph}(f)^{-1}H^{'H}(f)\Gamma_{dpc}(f)b_{N}$$

$$\rightarrow g_{T,\aleph}(f) = g_{U,\aleph}(f) + R_{\aleph}(f)^{-1}\tilde{d}(f,1)\frac{1-\aleph\tilde{d}^{H}(f,1)R_{\aleph}(f)^{-1}H^{'H}(f)\Gamma_{dpc}(f)b_{N}}{\tilde{d}^{H}(f,1)R_{\aleph}(f)^{-1}\tilde{d}(f,1)}$$

■