MATH 336 Presentation
Fixed Point and Steffensen's Acceleration Method

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Fixed Point Iteration Method

Definition

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Remark

Fixed point problems and root finding problems are infact equivalent.

- if p is a fixed point of the function g, then p is a root of the function

$$f(x) = [g(x) - x]h(x)$$

  [as long as $h(x) \in \mathbb{R}$]

- if p is a root of the function of f, then p is a fixed point of the function

$$g(x) = x - h(x)f(x)$$

  [as long as $h(x) \in \mathbb{R}$]

Fixed Point Iteration Method

Definition

Let U be a subset of a metric space X.
A function g:U $\rightarrow$X called **Lipschitz continuous** provided there exists a constant $\lambda \geq 0$ (called Lipschitz constant)
such that $\forall$ x,y$\in$U d(g(x),g(y))$\leq$d(x,y)
if $\lambda \in$[0,1], then g is called **contraction** (with contraction constant λ).

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Theorem (A Fixed Point Theorem)

Suppose $g : [a, b] \rightarrow [a, b]$ is continuous. Then g has a fixed point.

Fixed Point Iteration Method

Lemma

A contraction has at most one fixed point.

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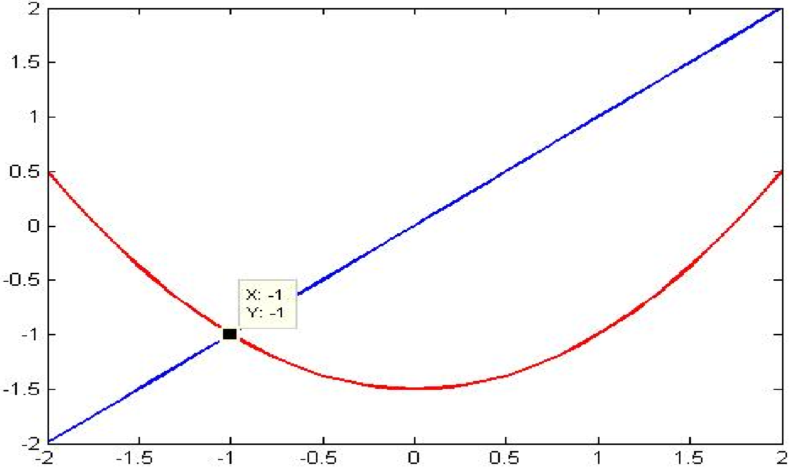
Corollary

Suppose $g : [a, b] \to [a, b]$ is continuous and $\lambda := sup \mid g'(x) \mid < 1$ for $x \in (a, b)$
Then g is a contraction with contraction constant λ.

Graphical determination of the existence of a fixed point for the function $g(x) = \frac{x^2-3}{2}$



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- *For any sequence $\{x_n\}$ defined by x_n=$g(x_{n-1})$, n=1,2,,... converges to this unique fixed point p.*

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$$| x_n - p | \leq \frac{\lambda^n}{1-\lambda} | x_1 - x_0 |$$

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  Moreover, we have the **a priori** *error estimate*

$$| \, x_n - p \, | \leq \frac{\lambda^n}{1-\lambda} \, | \, x_1 - x_0 \, |$$

and the **a posteriori** *error estimate*

$$| \, x_n - p \, | \leq \frac{\lambda}{1-\lambda} \, | \, x_n - x_{n-1} \, |$$

Banach Fixed Point Theorem

Proof

For $n > m$, we have

$$
\begin{aligned}
\mid x_n - x_m \mid &= \mid x_n - x_{n-1} + x_{n-1} - x_{n-2} + \ldots + x_{m+1} - x_m \mid \\
&\leq \mid x_n - x_{n-1} \mid + \mid x_{n-1} - x_{n-2} \mid + \ldots + \mid x_{m+1} - x_m \mid \\
&by* \\
&\leq (\lambda^{n-1} + \lambda^{n-2} + \ldots + \lambda^m) \mid x_1 - x_0 \mid \\
&= \lambda^m(\lambda^{n-m-1} + \lambda^{n-m-2} + \ldots + 1) \mid x_1 - x_0 \mid \\
&= \lambda^m \frac{1 - \lambda^{n-m}}{1 - \lambda} \mid x_1 - x_0 \mid \leq \frac{\lambda^m}{1 - \lambda} \mid x_1 - x_0 \mid
\end{aligned}
$$

so that x_n is Cauchy sequence in U.
Since U is complete, x_n converges to a point p $\in$ U

$* \mid x_k - x_{k-1} \mid = \mid g(x_{k-1}) - g(x_{k-2}) \mid \leq \lambda \mid x_{k-1} - x_{k-2} \mid \leq .. \leq \lambda^{k-1} \mid x_1 - x_0 \mid$

Continue.

Now,since g being contraction is continuous, we have

$$p = \lim_{n\to\infty} x_n = \lim_{n\to\infty} g(x_{n-1}) = g(lim_{n\to\infty} x_{n-1}) = g(p)$$

so that p is fixed point of g.
By the lemma p is the unique fixed point of g.

Continue.

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Since

$$\mid x_n - x_m \mid \leq \frac{\lambda^m}{1-\lambda} \mid x_1 - x_0 \mid,$$

letting n$\to \infty$

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we get

$$\mid p - x_m \mid \leq \frac{\lambda^m}{1-\lambda} \mid x_1 - x_0 \mid$$

for $y_0 = x_{n-1}$, $y_1 = x_n$

$$\mid y_1 - p \mid \leq \frac{\lambda}{1-\lambda} \mid y_1 - y_0 \mid$$

The Fixed Point Algorithm

The Fixed Point Algorithm

If g has a fixed point p, then the fixed point algorithm generates a sequence $\{x_n\}$ defined as
x_0: arbitrary but fixed,
$x_n = g(x_{n-1})$, n=1,2,3,... to approximate p.

Fixed Point The Case Where Multiple Derivatives Are Zero at The Fixed Point

Theorem

Let g be a continuous function on the closed internal $[a, b]$ with $\alpha > 1$ continuous derivatives on the internal (a,b).
Further, Let p $\in$(a,b) be a fixed point of g.

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then there exist a $\delta > 0$ such that for any $p_0 \in [p-\delta, p+\delta]$, the sequence $p_n = g(p_{n-1})$ converges to the fixed point p of order α with asymtotic error constant

$$\lim_{n\to\infty} \frac{| e_{n+1} |}{| e_n |^{\alpha}} = \frac{| g^{(\alpha)}(p) |}{\alpha!}$$

.

Proof

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Let $\lambda < 1$. Since $g'(p) = 0$ and g' is continuous, it follows that there exists a $\delta > 0$ such that $| g'(x) | \leq \lambda < 1$ for all $x \in I \equiv [p-\delta, p+\delta]$ From this, it follows that g:I→I; for if x∈ I then,

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$$\begin{aligned} \mid g(x) - p \mid &= \mid g(x) - g(p) \mid \\ &= \mid g'(\xi) \mid \mid x - p \mid \\ &\leq \lambda \mid x - p \mid < \mid x - p \mid \\ &\leq \delta \end{aligned}$$

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Therefore by the a fixed point theorem established earlier, the sequence $p_n = g(p_{n-1})$ converges to the fixed point p for any $p_0 \in [p-\delta, p+\delta]$.

Continue

To establish the order of convergence, let x∈ I and expand the iteration function g into a Taylor series about x=p:

$$g(x) = g(p) + g'(p)(x-p) + ... + \frac{g^{\alpha-1}(p)}{(\alpha-1)!}(x-p)^{\alpha-1} + \frac{g^{\alpha}(\xi)}{(\alpha)!}(x-p)^{\alpha}$$

where ξ is between x and p.

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Using the hypotheses regarding the value of $g^{(k)}(p)$ for $1 \leq k \leq \alpha - 1$ and letting $x = p_n$, the Taylor series expansion simplifies to

$$p_n + 1 - p = \frac{g^(\alpha)(\xi)}{\alpha!}(p_n - p)^{\alpha}$$

where ξ is now between p_n and p.

Continue.

The definitions of fixed point iteration scheme and of a fixed point have been used to replace $g(p_n)$ with p_{n+1} and g(p) with p.

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Finally, let $n \to \infty$.Then $p_n \to p$, forcing $\xi \to$ p also. Hence

$$\lim_{n\to\infty} \frac{\mid e_{n+1} \mid}{\mid e_n \mid^{\alpha}} = \frac{\mid g^{(\alpha)}(p) \mid}{\alpha!}$$

or $p_n \to$ p of order α.

□

Theorem (Aitken's Δ^2 method)

Suppose that

- $\{x_n\}$ *is a sequence with* $x_n \neq p$ *for all* $n \in \mathbb{N}$
- *there is a constant* $c \in \Re \setminus \{\mp 1\}$ *and a sequece* $\{\delta_n\}$ *such that*
  - $\lim_{n\to\infty} \delta_n = 0$
  - $x_{n+1} - p = (c + \delta_n)(x_n - p)$ *for all* $n \in \mathbb{N}$

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Then

- $\{x_n\}$ *converges to* p *iff* $|c| < 1$
- *if* $|c| < 1$ *, then*

$$y_n = \frac{x_{n+2}x_n - x_{n+1}^2}{x_{n+2} - 2x_{n+1} + x_n} = x_n - \frac{(x_{n+1} - x_n)^2}{x_{n+2} - 2x_{n+1} + x_n}$$

is well-defined for all sufficiently large n.

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Moreover $\{y_n\}$ *converges to p faster than* $\{x_n\}$ *in the sense that*

$$\lim_{n\to\infty} \frac{y_n - p}{x_n - p} = 0$$

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Let $e_n = x_n - p$

$\square$

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$y_n - p = \frac{(c+\delta_n)(\delta_{n+1}-\delta_n)}{(c-1)^+ c(\delta_{n+1}+\delta_n)+\delta_n(\delta_{n+1}-2)}$

Therefore $\lim_{n\to\infty} \frac{y_n - p}{x_n - p} = 0$

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Once we have x_0, x_1, and x_2, we can compute

$$y_0 = x_0 - \frac{(x_1 - x_0)^2}{x_2 - 2x_1 + x_0}$$

At this point we "restart" the fixed point iteration with $x_0 = y_0$

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e.g.

$$x_3 = y_0, x_4 = g(x_3), x_5 = g(x_4),$$

and compute

$$y_3 = x_3 - \frac{(x_4 - x_3)^2}{x_5 - 2x_4 + x_3}$$

Comparison with Fixed Point Iteration and Steffensen's Acceleration Method

EXAMPLE

Use the Fixed Point iteration method to find a solution to $f(x) = x^2 - 2x - 3$ using $x_0 = 0$, tolerance $=10^{-1}$ and compare the approximations with those given by Steffensen's Acceleration method with $x_0 = 0$, tolerance $= 10^{-2}$.

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- We can see that my MATLAB code while Fixed Point iteration method reaches the root by 788 iteration, Steffensen's Acceleration method reaches the root by only 3 iterations.